Liutex core line and objective Liutex

Presenter: Yifei Yu

Math department, the University of Texas at Arlington

Popular Vortex Structure display methods

1. **Streamline:**

Advantage:

- Consistent with people's intuition

Disadvantage:

- Velocity is a Galilean variant variable.
- The streamline bend is not obvious when the translation speed is large.
- Not reveal the rotation strength.

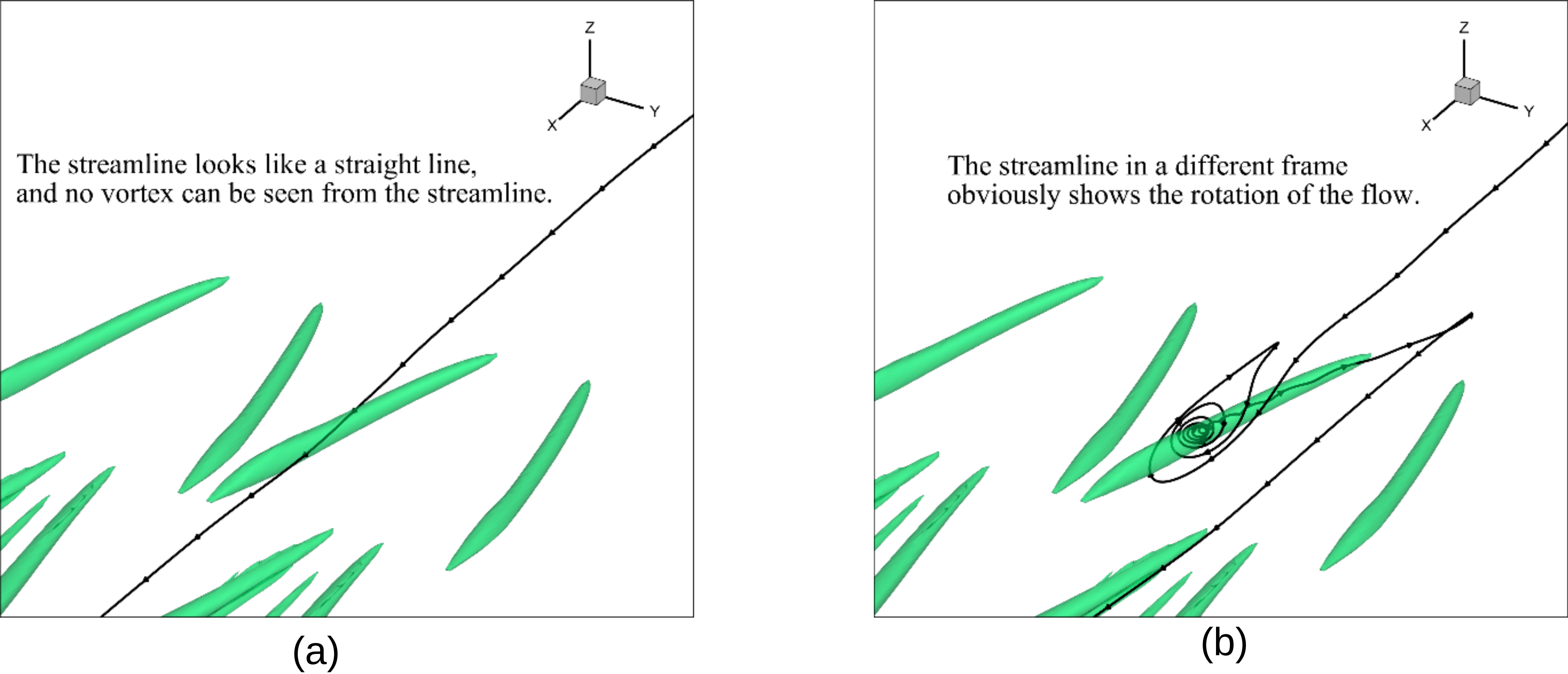


Fig.1 streamline in (a) natural frame(b) the frame that move at the same velocity of the selected point.

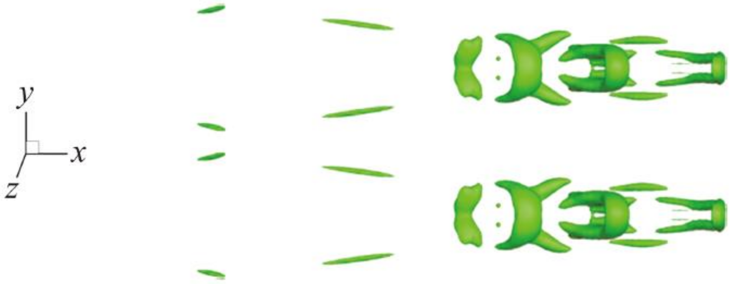
2. Iso-surface of vortex identification methods:

Advantage:

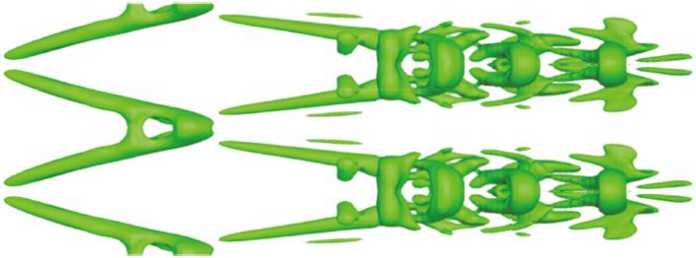
- Some vortex identification methods are Galilean invariant.
- Vortex strength can be known from the value of the vortex identification methods.

Disadvantage:

- The choice of threshold does not have a standard for most methods.
- The vortex structure is not unique.
- Not know how the vortex rotates. (rotational axis)



(a)



(b)

Fig.2 (a) vortex breakdown when choose a big threshold
(b) vortex not break down when choose a small threshold.

Requirement for a good vortex structure display

- Unique
- Reveal how the vortex rotates (rotational axis)
- Reveal the rotation strength
- Galilean invariant (objective is better)
- ……

Vortex structure is fluid macroscopic performance. To find a good vortex structure method, a correct local fluid rotation indicator is required.

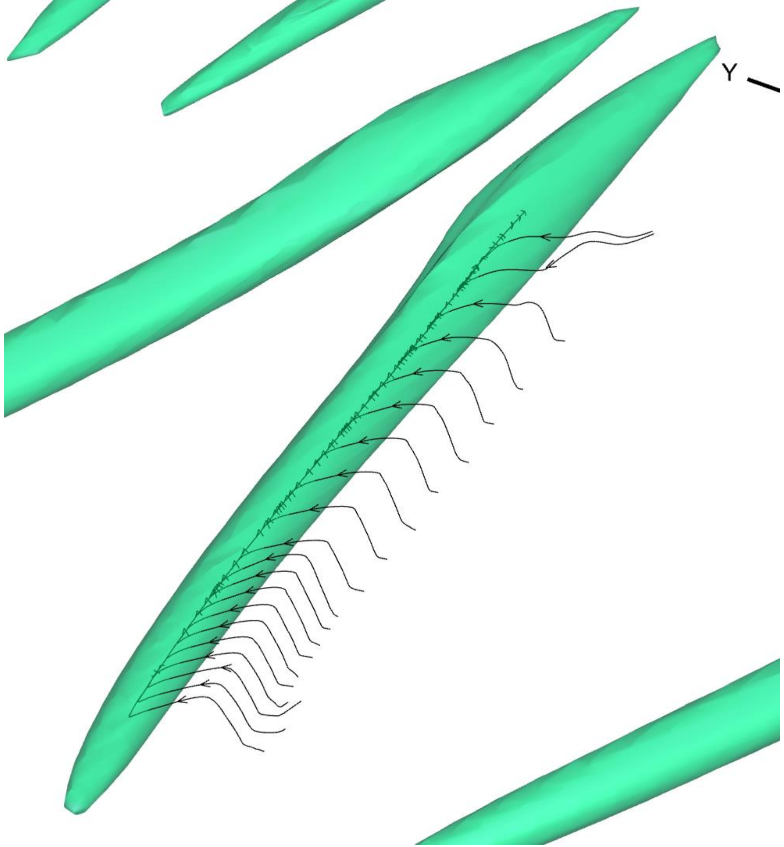
Liutex

Definition:

Liutex vector $\vec{R} = R\vec{r}$ is a vector s.t.

(1) $\vec{r}$ is the unit real eigenvector of the velocity gradient tensor ∇R with $\vec{\omega} \cdot \vec{r} > 0$ where $\vec{\omega}$ is the vorticity.

(2) $R = \vec{\omega} \cdot \vec{r} - \sqrt{(\vec{\omega} \cdot \vec{r})^2 - 4\lambda_{ci}^2}$ where λ_{ci} is the imaginary part of the complex eigenvalue of ∇R



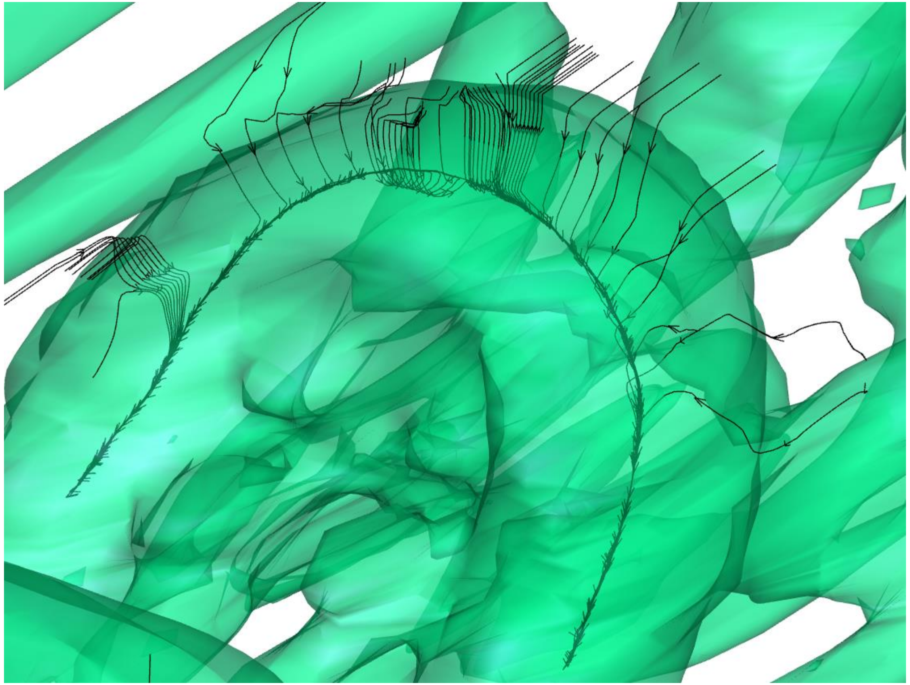


Fig.3 R=0.07 iso-surface with Liutex gradient lines.

Some observations

- Liutex gradient lines converge to a center line and this center line is unique.
- This center line is the rotational axis because it is parallel to the Liutex direction at the same location.
- The Liutex magnitudes at the center line positions reveal the rotation strength.
- Liutex gradient line is Galilean invariant as Liutex is Galilean invariant.
- ……

The center line is a good way to display vortex structure.

Liutex core line

Definition:

The vortex core center or vortex rotation axis line is defined as a line consisting of points which satisfy the condition

$$\nabla R \times \vec{r} = 0, \qquad R > 0 \tag{1}$$

where $\vec{r}$ represents the direction of Liutex vector.

Explanation:

∇R is perpendicular to the iso-surface except the iso-surface degenerates to a line(point). The definition essentially means find the point whose ∇R is parallel to the iso-surface.

How to realize the Liutex core lines

Method 1(in progress):
Draw many Liutex gradient lines and remove “hairs”

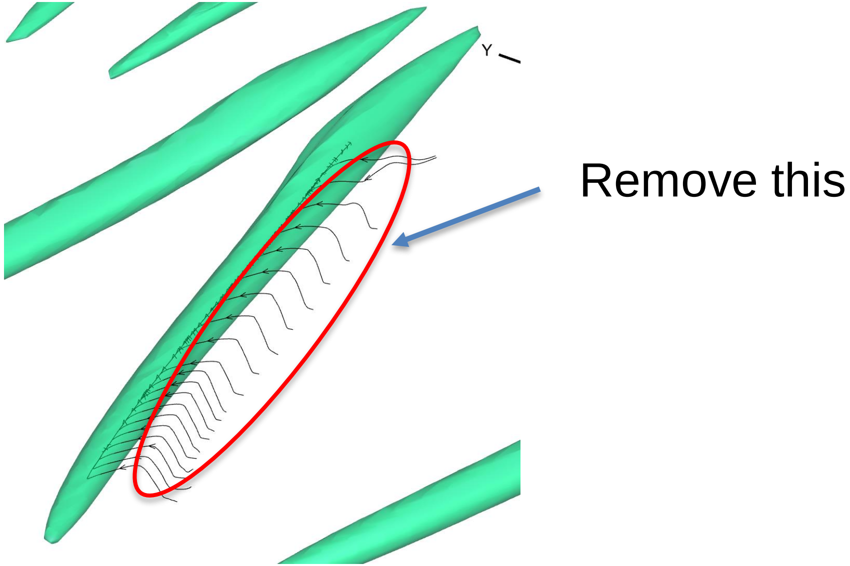


Fig.4 R=0.07 iso-surface with Liutex gradient lines.

Method 2:
Find seed points and draw Liutex lines.

Advantage:
No need to remove “hair” and can draw the structure of one whole vortex at one time.

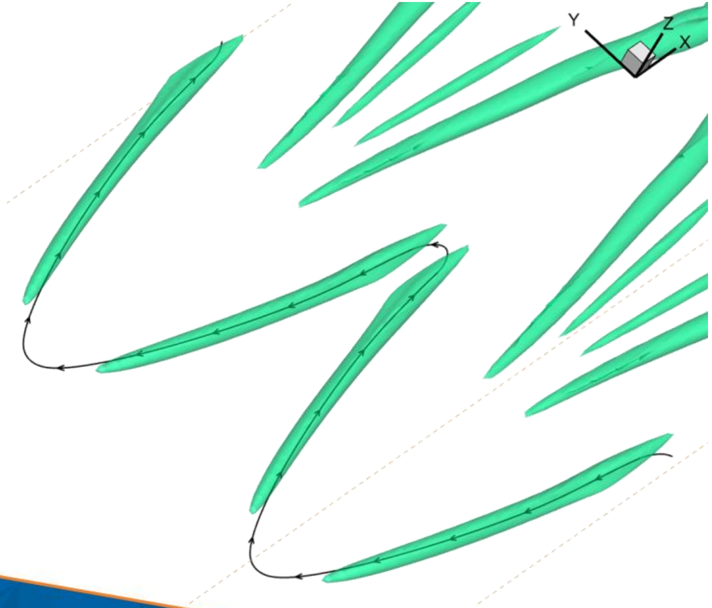
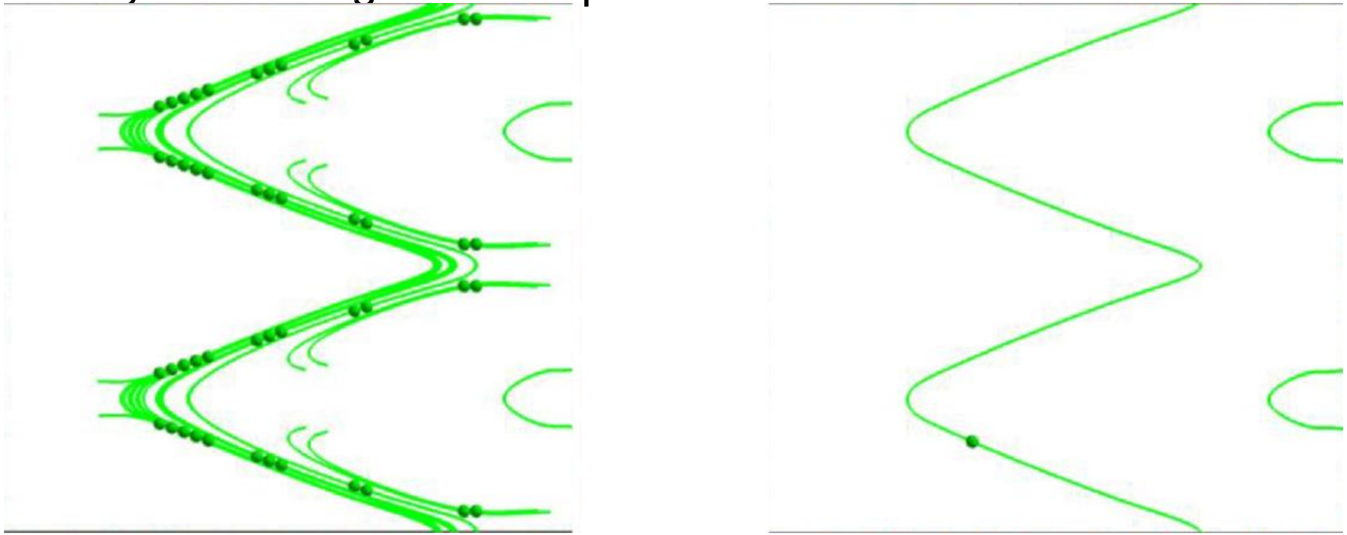


Fig.5 Liutex Line drawn by one seed point.

Disadvantage:
The program may find more than one seed point for one vortex(fake points) and need to find ways to distinguish fake points. Li et.al.* did contribution on this aspect.



(a) (b)

Fig.6* Find seed points (a) including fake points (b) after removing fake points.

*Li, H., Liu, Y., Wang, D., & Xu, H. (2022). Liutex (vortex) core and tube identification and automatic generation algorithms. Computers & Fluids, 105731.

More discussion:

The numerical result shows gradient center line does not exactly overlap the Liutex line. It has not been mathematically proved that gradient center line exactly overlaps Liutex line. This discrepancy may cause by numerical errors or the two lines are simply different.

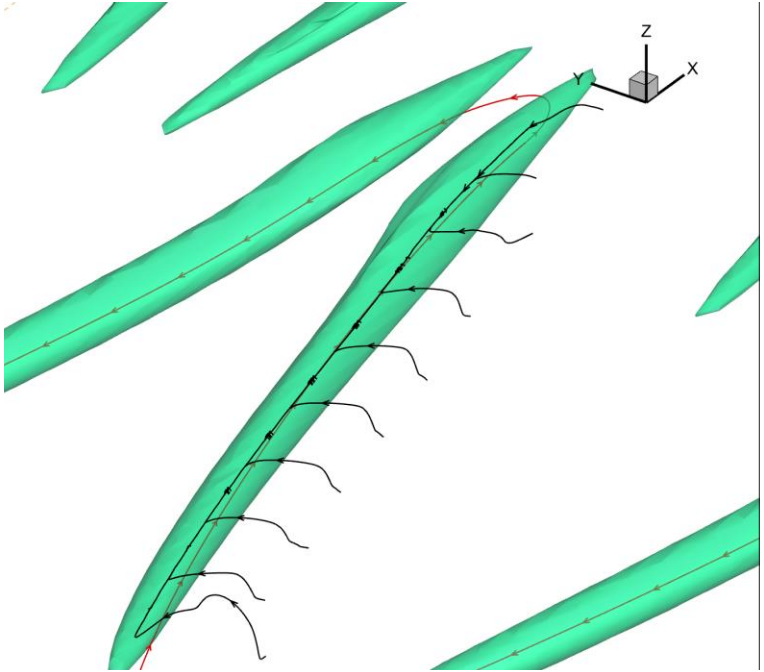


Fig.7 Liutex line(red) and Liutex gradient line(black)

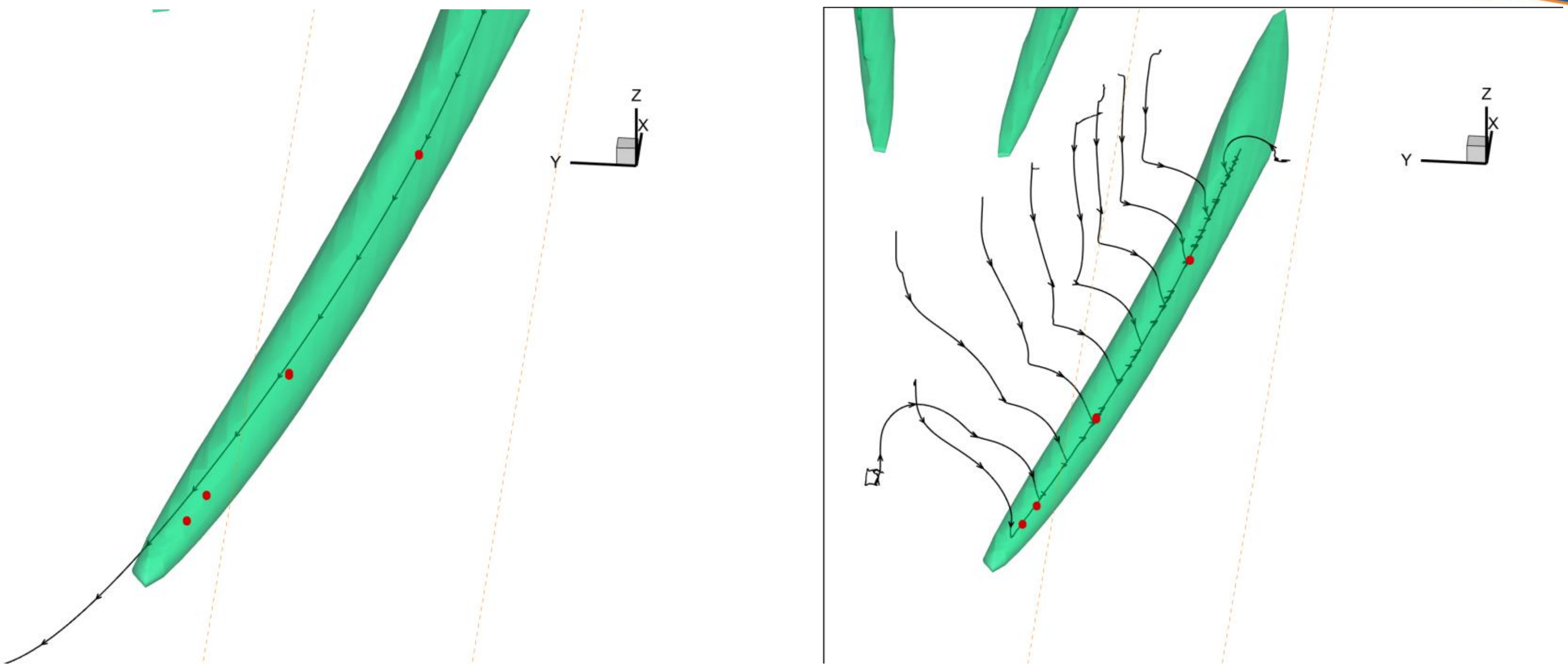


Fig.8 Seed points(red points), Liutex Line(left) and Liutex magnitude line(right)

Objective Liutex

Even though Liutex is Galilean invariant, it is not objective which means the vortex structure shown by Liutex can be different for different non-inertial observers.

We would like to find a way through which we can obtain the same vortex structure for all observers.

Liutex is Galilean invariant

→

Liutex structure is the same in different inertial frames

→

If we can find Liutex in any one inertial frame from the data collected by non-inertial observer, this vortex structure is objective.

Upper case letters are used to express variables in the inertial coordinate system.
Lower case letters are used to express variables in the non-inertial coordinate system.

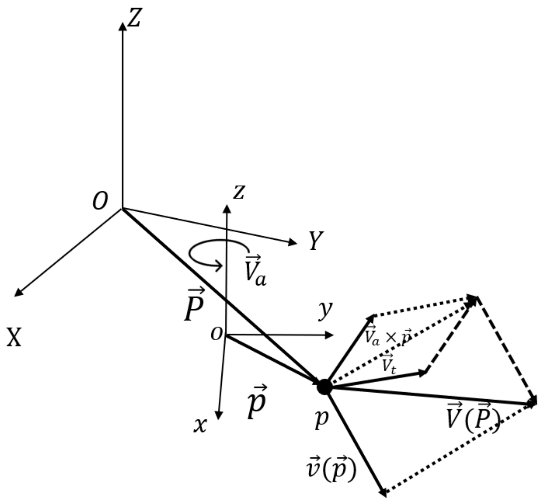


Fig. 9 relation between velocity in the inertial coordinate system and velocity in the non-inertial coordinate system of the same point

The relation between velocity in the inertial coordinate system and velocity in the non-inertial coordinate system of the same point is

$$\vec{V}(\vec{P}) = \vec{v}(\vec{p}) + \vec{V}_t(t) + \vec{V}_a(t) \times \vec{p} \tag{2}$$

Since our purpose is to find the velocity gradient tensor in any one inertial coordinate system, a special one is chosen for simplification, i.e., the origin point and X,Y,Z axes of the special inertial coordinate system are the same as the non-inertial coordinate system, and the special inertial coordinate system is doing the same translation motion as the non-inertial one. The only difference between is two coordinate systems is the non-inertial one can have acceleration and do rotation while the inertial one cannot. In this situation, $\vec{V}_t = 0$ and $\vec{P} = \vec{p}$. Thus, Eq. 2 becomes

$$\vec{V}(\vec{P}) = \vec{v}(\vec{P}) + \vec{V}_a(t) \times \vec{P} \qquad (3)$$

where $\vec{V}_a = V_{ax}\vec{X} + V_{ay}\vec{Y} + V_{az}\vec{Z}$ is the angular velocity of the observer measured in the inertial coordinate.

Or

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} V_{ay}Z - V_{az}y \\ V_{az}X - V_{ax}Z \\ V_{ax}Y - V_{ay}X \end{bmatrix} \qquad (4)$$

Take partial derivative with respect to $X(x)$

$$\begin{bmatrix} \frac{\partial U}{\partial X} \\ \frac{\partial V}{\partial X} \\ \frac{\partial W}{\partial X} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 \\ V_{az} \\ -V_{ay} \end{bmatrix} \tag{5}$$

Similarly,

$$\begin{bmatrix} \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial Y} \\ \frac{\partial W}{\partial Y} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial y} \end{bmatrix} + \begin{bmatrix} -V_{az} \\ 0 \\ V_{ax} \end{bmatrix} \qquad \begin{bmatrix} \frac{\partial U}{\partial Z} \\ \frac{\partial V}{\partial Z} \\ \frac{\partial W}{\partial Z} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial z} \end{bmatrix} + \begin{bmatrix} V_{ay} \\ -V_{ax} \\ 0 \end{bmatrix} \tag{6}$$

So,

$$\begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} & \frac{\partial U}{\partial Z} \\ \frac{\partial Y}{\partial X} & \frac{\partial Y}{\partial Y} & \frac{\partial Y}{\partial Z} \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} & \frac{\partial W}{\partial Z} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} + \begin{bmatrix} 0 & -V_{az} & V_{ay} \\ V_{az} & 0 & -V_{ax} \\ -V_{ay} & V_{ax} & 0 \end{bmatrix} \qquad (7)$$

$V_{ax}, V_{ay}\ and\ V_{az}$ are independent on X, Y, Z which means they are the same throughout the whole space. In another word, if we can find out $V_{ax}, V_{ay}\ and\ V_{az}$ at one point, these values can be used for the whole space.

Next question is how to find V_{ax}, V_{ay} *and* V_{az}

Theorem: If $\vec{P}$ is a point in the inertial XYZ s.t.

(1) Vorticity of $\vec{P}$ measured in XYZ is zero
(2) The XYZ coordinate system has the same origin and basis of the observer's xyz coordinate system.
(3) The XYZ coordinate system is doing the same translation motion as the observer's xyz coordinate system.

Then, $V_{ax}\vec{x} + V_{ay}\vec{y} + V_{az}\vec{z} = -\frac{1}{2}(\omega_x\vec{x} + \omega_y\vec{y} + \omega_z\vec{z})$ where $\omega_x\vec{x} + \omega_y\vec{y} + \omega_z\vec{z}$ is the vorticity of the point $\vec{p}$ corresponding to $\vec{P}$ under the observer's coordinate.

Proof:

$$\nabla \vec{V} = A + B \tag{8}$$

$$A = \frac{1}{2}\left[\nabla \vec{V} + \left(\nabla \vec{V}\right)^T\right] \qquad B = \frac{1}{2}\left[\nabla \vec{V} - \left(\nabla \vec{V}\right)^T\right] \tag{9}$$

$$A + B = \nabla \vec{v} + \begin{bmatrix} 0 & -V_{az} & V_{ay} \\ V_{az} & 0 & -V_{ax} \\ -V_{ay} & V_{ax} & 0 \end{bmatrix} \tag{10}$$

$$\nabla \vec{v} = A - \begin{bmatrix} 0 & -V_{az} & V_{ay} \\ V_{az} & 0 & -V_{ax} \\ -V_{ay} & V_{ax} & 0 \end{bmatrix} = A + \begin{bmatrix} 0 & V_{az} & -V_{ay} \\ -V_{az} & 0 & V_{ax} \\ V_{ay} & -V_{ax} & 0 \end{bmatrix} \tag{11}$$

Obviously, $\begin{bmatrix} 0 & V_{az} & -V_{ay} \\ -V_{az} & 0 & V_{ax} \\ V_{ay} & -V_{ax} & 0 \end{bmatrix}$ is the vorticity matrix measured in the observer's coordinate since A is a symmetric matrix.

Therefore,

$$\left(V_{ax}, V_{ay}, V_{az}\right) = -\frac{1}{2}\left(\omega_x, \omega_y, \omega_z\right) \tag{12}$$

Steps to obtain objective vortex structure:

1. Pick a point with zero vorticity measured in the inertial frame. The point can be selected based on physical properties of the flow e.g., points in the inviscid region.
2. Calculate vorticity at the selected point. Then $\left(V_{ax}, V_{ay}, V_{az}\right) = -\frac{1}{2}\left(\omega_x, \omega_y, \omega_z\right)$.
3. The velocity gradient tensor of all points in an inertial coordinate can be obtained from

$$\nabla\vec{V} = \nabla\vec{v} + \begin{bmatrix} 0 & -V_{az} & V_{ay} \\ V_{az} & 0 & -V_{ax} \\ -V_{ay} & V_{ax} & 0 \end{bmatrix} \tag{13}$$

4. Calculate Liutex from $\nabla\vec{V}$.

Thank you for your attention!