UNIVERSITY OF TEXAS ARLINGTON

Liutex Based New Fluid Kinematics

Lecture 4 of Liutex Short Course

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Outlines

I. Classical Fluid Kinematics
II. Principal Decomposition of Velocity Gradient Tensor in Principal Coordinates
III. New Fluid Kinematic
IV. Divergence Relation between Velocity Gradient Tensor, Strain and Vorticity (Liu 2021)
V. Symmetric Strain-Based Navier Stokes Equations
VI. Vorticity-Based New Governing Equations (Liu & Liu 2021)
VII. Conclusions

1. Helmholtz (Cauchy-Stokes Tensor) Decomposition

Helmholtz Decomposition (Wiki: https://en.wikipedia.org/wiki/Helmholtz_decomposition)

The Helmholtz decomposition states that a vector field (satisfying appropriate smoothness and decay conditions) can be decomposed as the sum of the form $-\varphi + \nabla \times \vec{A}$ where φ is a scalar field called "scalar potential", and $\vec{A}$ is a vector field, called a vector potential. (Or potential part and vorticity part –misunderstood as rotation)

In fluid dynamic textbooks: Helmholtz fluid particle velocity decomposition is $\vec{v} = Translation + Deformation + Rotation$ which is equivalent to Cauchy-Stokes Tensor Decomposition$\nabla\vec{\boldsymbol{v}}$= $\boldsymbol{A}$+ $\boldsymbol{B}$ (Vorticity $\nabla \times \vec{v}$ is considered as rotation)

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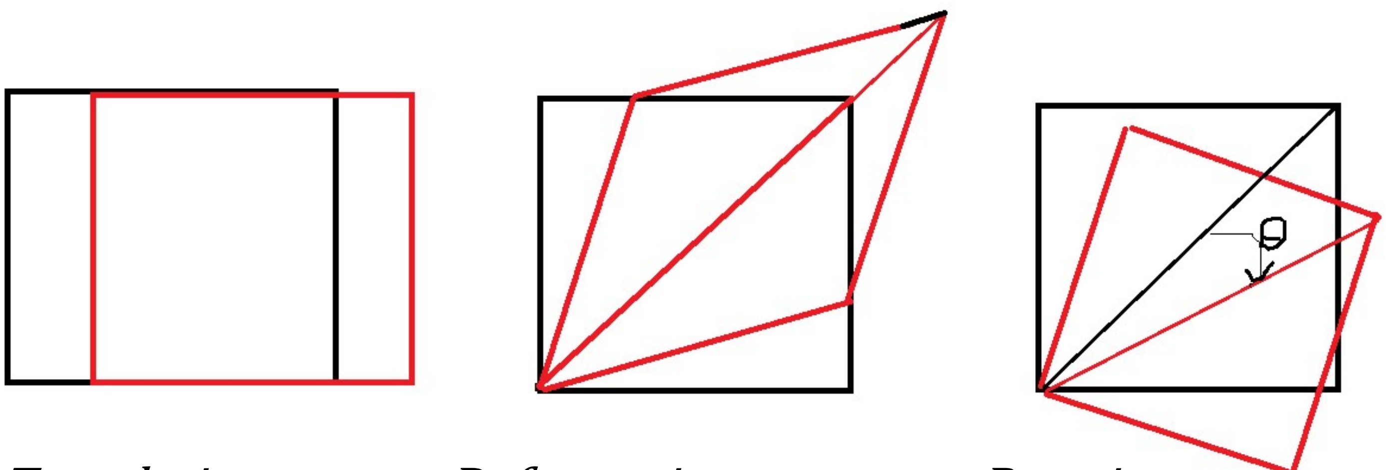


Translation *Deformation* *Rotation*

1. Helmholtz (Cauchy-Stokes Tensor) Decomposition

In classical fluid kinematics, the the velocity gradient tensor is decomposed into a symmetric part and an anti-symmetric part

$$\nabla\vec{\boldsymbol{v}} = \boldsymbol{A} + \boldsymbol{B}$$

$$\boldsymbol{A} = \frac{1}{2}\left(\nabla\vec{v} + \nabla\vec{v}^{\mathrm{T}}\right) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

$$\boldsymbol{B} = \frac{1}{2}\left(\nabla\vec{v} - \nabla\vec{v}^{\mathrm{T}}\right) = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) \\ -\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ -\frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & -\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix}$$

Traditionally, people think ***A*** represents deformation and ***B*** for rotation – Unfortunately, NO

Liutex – Mathematical Definition of Vortex

Liutex represent fluid rotation, but not vorticity

$\vec{\boldsymbol{R}} = R\vec{\boldsymbol{r}}$

$\vec{r}$ is real eigenvector of $(\nabla\vec{\boldsymbol{v}})^{\boldsymbol{T}}$

$R = \langle\vec{\omega}, \vec{r}\rangle - \sqrt{\langle\vec{\omega}, \vec{r}\rangle^2 - 4\lambda_{ci}^2}$ (Wang, Gao, Liu, 2019)

where $\vec{\omega}$ is the vorticity vector, $\vec{r}$ is the Liutex direction vector, and λ_{ci} is the imaginary part of the complex eigenvalue of $\nabla\vec{\boldsymbol{v}}$.

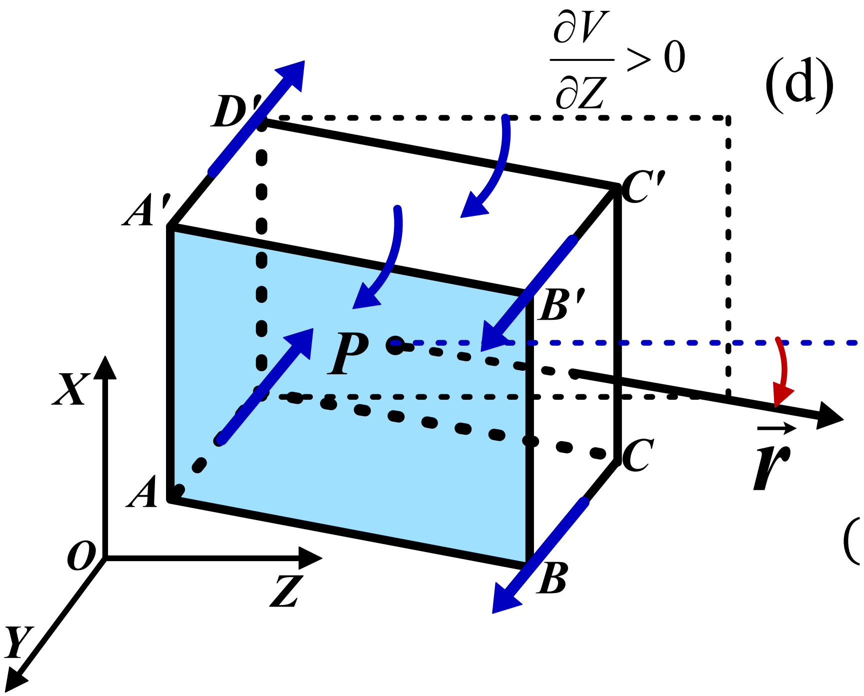
Problems with Cauchy-Stokes (CS) Decomposition

(1) Since vorticity cannot represent fluid rotation, the vorticity tensor is a mixture of rigid rotation and non-rotating anti-symmetric shear.

(2) The symmetric strain-rate tensor is a mixture of stretching and symmetric shear (diagonal and off-diagonal elements are dependent on coordinates)

(3) The CS decomposition is dependent on the selection of coordinate system and is therefore not invariant (stretch is mixed with shear)

2. Principal Coordinate System and Principal Matrix



$$\nabla\vec{V} = Q^T \nabla\vec{v} Q \quad \textit{to make } \vec{Z} \textit{ parallel to } \vec{r}$$

$$Q^T \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ r_x & r_y & r_z \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(\nabla\vec{V})^T = \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} & 0 \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} & 0 \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} & \frac{\partial W}{\partial Z} \end{bmatrix} = \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} & 0 \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} & 0 \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} & \lambda_r \end{bmatrix}$$

$$(\nabla\vec{V}_\theta)^T = \begin{bmatrix} \lambda_{cr} & -\frac{1}{2}R & 0 \\ \frac{1}{2}R + \epsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix}$$ - Principal Matrix

Tensor is unique but matrix is dependent on coordinates and not unique. Try to find a unique coordinate system and then unique matrix – Principal Matrix

2. Tensor Principal Decomposition in Principal Coordinates (Unique)

Principal tensor matrix

(We need a unique matrix for the velocity gradient tensor)

$$(\nabla\vec{V})^T = \begin{bmatrix} \lambda_{cr} & -\frac{R}{2} & 0 \\ \frac{R}{2}+\epsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix}$$ The principal tensor matrix should be $\nabla\vec{V} = \begin{bmatrix} \lambda_{cr} & \frac{R}{2}+\epsilon & \xi \\ -\frac{R}{2} & \lambda_{cr} & \eta \\ 0 & 0 & \lambda_r \end{bmatrix}$

Principal decomposition

$$\nabla\vec{V} = \begin{bmatrix} \lambda_{cr} & \frac{R}{2}+\epsilon & \xi \\ -\frac{R}{2} & \lambda_{cr} & \eta \\ 0 & 0 & \lambda_r \end{bmatrix} = \begin{bmatrix} 0 & R/2 & 0 \\ -R/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_r \end{bmatrix} + \begin{bmatrix} 0 & \in & \xi \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{bmatrix} = \boldsymbol{R} + \boldsymbol{SC}\text{+S}$$

Shear is not symmetric!

Our previous decomposition

$$(\nabla\vec{V})^T = \begin{bmatrix} 0 & -R/2 & 0 \\ R/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_r \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \in & 0 & 0 \\ \xi & \eta & 0 \end{bmatrix} = -\boldsymbol{R} + \boldsymbol{SC}\text{+}\boldsymbol{S}^{\boldsymbol{T}}$$

2. Tensor Principal Decomposition in Principal Coordinates (Unique)

Principal decomposition

$$\nabla\vec{V} = \begin{bmatrix} \lambda_{cr} & \frac{R}{2}+\epsilon & \xi \\ -\frac{R}{2} & \lambda_{cr} & \eta \\ 0 & 0 & \lambda_r \end{bmatrix} = \begin{bmatrix} 0 & R/2 & 0 \\ -R/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_r \end{bmatrix} + \begin{bmatrix} 0 & \in & \xi \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \boldsymbol{R} + \boldsymbol{SC} + \mathrm{S}$$

$\nabla\vec{\boldsymbol{V}}$ = Rotation (Liutex)+Stretching(Compression)+Shear

It is unique and physical meaning is very clear.

Totally different from Cauchy-Stokes decomposition

Koloar 2007 and Li et al. 2014 have similar ideas to decompose the velocity gradient tensor

Kolář, V., Vortex identification: New requirements and limitations [J]. International Journal of Heat and Fluid Flow, (2007), 28(4): 638-652.

Li, Z., Zhang, X., He., F., Evaluation of vortex criteria by virtue of the quadruple decomposition of velocity gradient tensor. Acta Physics Sinica, 2014, 63(5): 054704, in Chinese

3. New Fluid Kinematics

Principal coordinates are different at each point, and we must come back to the original xyz coordinate system:

$$\nabla\vec{\boldsymbol{v}} = \boldsymbol{QP}(\boldsymbol{R})\boldsymbol{P}^T\boldsymbol{Q}^T + \boldsymbol{QP}(\boldsymbol{S})\boldsymbol{P}^T\boldsymbol{Q}^T + \boldsymbol{QP}(\boldsymbol{SC})\boldsymbol{P}^T\boldsymbol{Q}^T = \widetilde{\boldsymbol{R}} + \widetilde{\boldsymbol{S}} + \widetilde{\boldsymbol{SC}}$$

$$\text{Assume } \boldsymbol{QP} = \begin{bmatrix} Q_{11} & Q_{12} & r_x \\ Q_{21} & Q_{22} & r_y \\ Q_{31} & Q_{32} & r_z \end{bmatrix}$$

$$\boldsymbol{P}^T\boldsymbol{Q}^T \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ r_x & r_y & r_z \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

3. New Fluid Kinematics

Transfer $\vec{\boldsymbol{R}}$ bask to the original xyz-system

In the principal coordinate, $\boldsymbol{R} = \begin{bmatrix} 0 & \frac{R}{2} & 0 \\ -\frac{R}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In the original xyz coordinate,

$$\widetilde{\boldsymbol{R}} = \boldsymbol{QP}(\boldsymbol{R})\boldsymbol{P}^T\boldsymbol{Q}^T = \begin{bmatrix} 0 & \frac{R_z}{2} & -\frac{R_y}{2} \\ -\frac{R_z}{2} & 0 & \frac{R_x}{2} \\ \frac{R_y}{2} & -\frac{R_x}{2} & 0 \end{bmatrix}$$

$\vec{\boldsymbol{R}}$ is Galilean invariant and easy to transfer

3. New Fluid Kinematics

Transfer SC **(Stretch and Compression) back to the xyz system**

In the original xyz coordinate, the stretching (compression) part can be obtained by anti-***QP*** rotation

$$\widetilde{\boldsymbol{SC}} = \boldsymbol{QP}(\boldsymbol{SC})\boldsymbol{P}^T\boldsymbol{Q}^T = \boldsymbol{QP}\begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_r \end{bmatrix}\boldsymbol{P}^T\boldsymbol{Q}^T =$$

$$\boldsymbol{QP}\left\{\begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_{cr} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_r - \lambda_{cr} \end{bmatrix}\right\}\boldsymbol{P}^T\boldsymbol{Q}^T$$

$$= \lambda_{cr}\boldsymbol{QP}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\boldsymbol{P}^T\boldsymbol{Q}^T + \begin{bmatrix} Q_{11} & Q_{12} & r_x \\ Q_{21} & Q_{22} & r_y \\ Q_{31} & Q_{32} & r_z \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_r - \lambda_{cr} \end{bmatrix}\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ r_x & r_y & r_z \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_{cr} \end{bmatrix} + (\lambda_r - \lambda_{cr})\begin{bmatrix} Q_{11} & Q_{12} & r_x \\ Q_{21} & Q_{22} & r_y \\ Q_{31} & Q_{32} & r_z \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_x & r_y & r_z \end{bmatrix} = \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_{cr} \end{bmatrix} + (\lambda_r - \lambda_{cr})\begin{bmatrix} {r_x}^2 & r_x r_y & r_x r_z \\ r_y r_x & {r_y}^2 & r_y r_z \\ r_z r_x & r_z r_y & {r_z}^2 \end{bmatrix}$$

Because $\boldsymbol{QP} = \begin{bmatrix} Q_{11} & Q_{12} & r_x \\ Q_{21} & Q_{22} & r_y \\ Q_{31} & Q_{32} & r_z \end{bmatrix}$ and $\boldsymbol{P}^T\boldsymbol{Q}^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ r_x & r_y & r_z \end{bmatrix}$, Q_1, Q_2, Q_3 *are orthogonal*

3. New Fluid Kinematics

$$\nabla\vec{v} = \widetilde{R} + \widetilde{S} + \widetilde{SC}$$

$$\widetilde{S} = \nabla\vec{v} - \widetilde{R} - \widetilde{SC}$$

The shear part of the velocity gradient tensor is

$$\widetilde{S} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} - \begin{bmatrix} 0 & \frac{R_z}{2} & -\frac{R_y}{2} \\ -\frac{R_z}{2} & 0 & \frac{R_x}{2} \\ \frac{R_y}{2} & -\frac{R_x}{2} & 0 \end{bmatrix} - \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_{cr} \end{bmatrix} - (\lambda_r - \lambda_{cr}) \begin{bmatrix} {r_x}^2 & r_x r_y & r_x r_z \\ r_y r_x & {r_y}^2 & r_y r_z \\ r_z r_x & r_z r_y & {r_z}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} - \frac{R_z}{2} & \frac{\partial w}{\partial x} + \frac{R_y}{2} \\ \frac{\partial u}{\partial y} + \frac{R_z}{2} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} - \frac{R_x}{2} \\ \frac{\partial u}{\partial z} - \frac{R_y}{2} & \frac{\partial v}{\partial z} + \frac{R_x}{2} & \frac{\partial w}{\partial z} \end{bmatrix} - \begin{bmatrix} \lambda_{cr} + (\lambda_r - \lambda_{cr}){r_x}^2 & (\lambda_r - \lambda_{cr}) r_x r_y & (\lambda_r - \lambda_{cr}) r_x r_z \\ (\lambda_r - \lambda_{cr}) r_y r_x & \lambda_{cr} + (\lambda_r - \lambda_{cr}){r_y}^2 & (\lambda_r - \lambda_{cr}) r_y r_z \\ (\lambda_r - \lambda_{cr}) r_z r_x & (\lambda_r - \lambda_{cr}) r_z r_y & \lambda_{cr} + (\lambda_r - \lambda_{cr}){r_z}^2 \end{bmatrix}$$

If $\widetilde{R} = 0$, $\widetilde{S} = \nabla\vec{v} - \widetilde{SC}$ - Non-vortex areas $\widetilde{S} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} - \begin{bmatrix} \frac{1}{3}tr & 0 & 0 \\ 0 & \frac{1}{3}tr & 0 \\ 0 & 0 & \frac{1}{3}tr \end{bmatrix}$

3. New Fluid Kinematics

Principal coordinates are different at each point and we must come back to the original xyz coordinate system:

$$\nabla\vec{v} = QP(R)P^TQ^T + QP(S)P^TQ^T + QP(SC)P^TQ^T = \widetilde{R} + \widetilde{S} + \widetilde{SC}$$

$$\widetilde{R} = Q(R)Q^T = \begin{bmatrix} 0 & \frac{R_z}{2} & -\frac{R_y}{2} \\ -\frac{R_z}{2} & 0 & \frac{R_x}{2} \\ \frac{R_y}{2} & -\frac{R_x}{2} & 0 \end{bmatrix}$$

$$\widetilde{SC} = \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_{cr} \end{bmatrix} + (\lambda_r - \lambda_{cr}) \begin{bmatrix} {r_x}^2 & r_x r_y & r_x r_z \\ r_y r_x & {r_y}^2 & r_y r_z \\ r_z r_x & r_z r_y & {r_z}^2 \end{bmatrix}$$

$$\widetilde{S} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} - R_z/2 & \frac{\partial w}{\partial x} + R_y/2 \\ \frac{\partial u}{\partial y} + R_z/2 & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} - R_x/2 \\ \frac{\partial u}{\partial z} - R_y/2 & \frac{\partial v}{\partial z} + R_x/2 & \frac{\partial w}{\partial z} \end{bmatrix} - \begin{bmatrix} \lambda_{cr} + (\lambda_r - \lambda_{cr}){r_x}^2 & (\lambda_r - \lambda_{cr})r_x r_y & (\lambda_r - \lambda_{cr})r_x r_z \\ (\lambda_r - \lambda_{cr})r_y r_x & \lambda_{cr} + (\lambda_r - \lambda_{cr}){r_y}^2 & (\lambda_r - \lambda_{cr})r_y r_z \\ (\lambda_r - \lambda_{cr})r_z r_x & (\lambda_r - \lambda_{cr})r_z r_y & \lambda_{cr} + (\lambda_r - \lambda_{cr}){r_z}^2 \end{bmatrix}$$

See our paper "New Fluid Kinematics", JHD, Springer Online First, https://link.springer.com/journal/42241 http://www.jhydrodynamics.com Journal of Hydrodynamics, May 11, 2021 https://doi.org/10.1007/s42241-021-0037-5

Vorticity Decomposition $\nabla \times \vec{V} = \vec{R} + \vec{S}$

Vorticity Decomposition can be done by vorticity tensor decomposition in principal coordinates:

$$\boldsymbol{B} = \begin{bmatrix} 0 & \frac{R}{2} + \frac{\epsilon}{2} & \frac{\xi}{2} \\ -\frac{R}{2} - \frac{\epsilon}{2} & 0 & \frac{\eta}{2} \\ -\frac{\xi}{2} & -\frac{\eta}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{R}{2} & 0 \\ -\frac{R}{2} & 0 & 0 \\ o & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{\epsilon}{2} & \frac{\xi}{2} \\ -\frac{\epsilon}{2} & 0 & \frac{\eta}{2} \\ -\frac{\xi}{2} & -\frac{\eta}{2} & 0 \end{bmatrix} = \boldsymbol{R} + \boldsymbol{VS},\ \boldsymbol{both\ anti - symmetric}$$

$$B \cdot d\vec{l} = d\vec{l} \times (\nabla \times \vec{V}) = d\bar{l} \times \vec{R} + d\vec{l} \times \vec{S} = d\vec{l} \times (\vec{R} + \vec{S})$$

Because $d\,\vec{l}$ is arbitrarily selected, then we have

$$\nabla \times \vec{V} = \vec{R} + \vec{S}$$

The vorticity vector is not vortex vector and must be decomposed to vortex vector (Liutex) and non-rotational anti-symmetric shear.

$\boldsymbol{\nabla} \times \vec{\boldsymbol{v}} = \vec{\boldsymbol{R}} + \vec{\boldsymbol{S}}$ in original xyz coordinates since both vorticity and Liutex are Galilean invariant

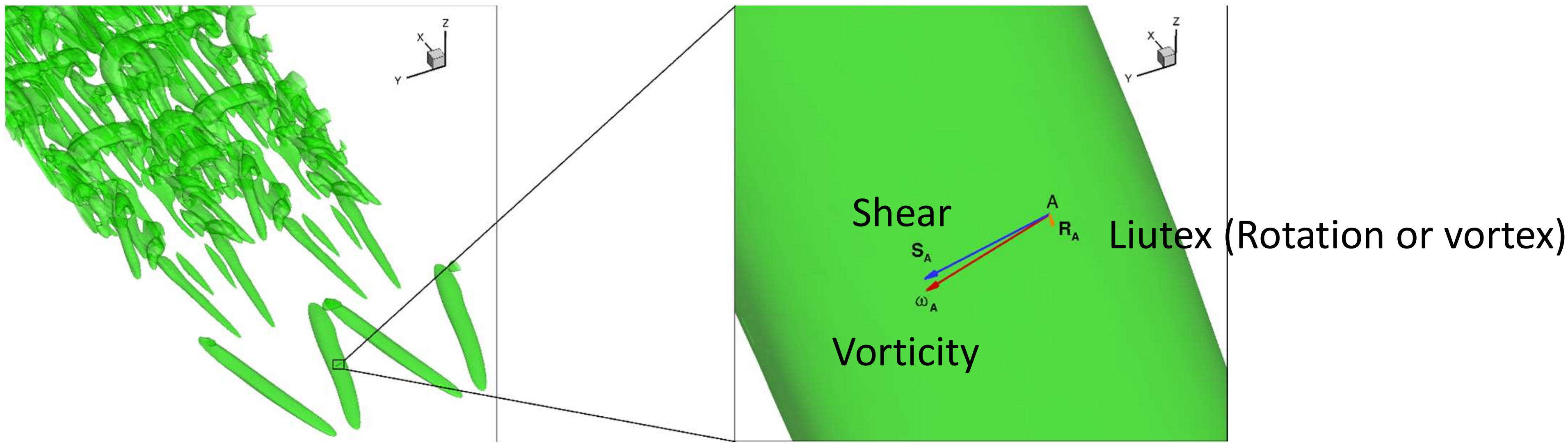


Fig. 3 Illustration of vorticity vector decomposition of Point A.

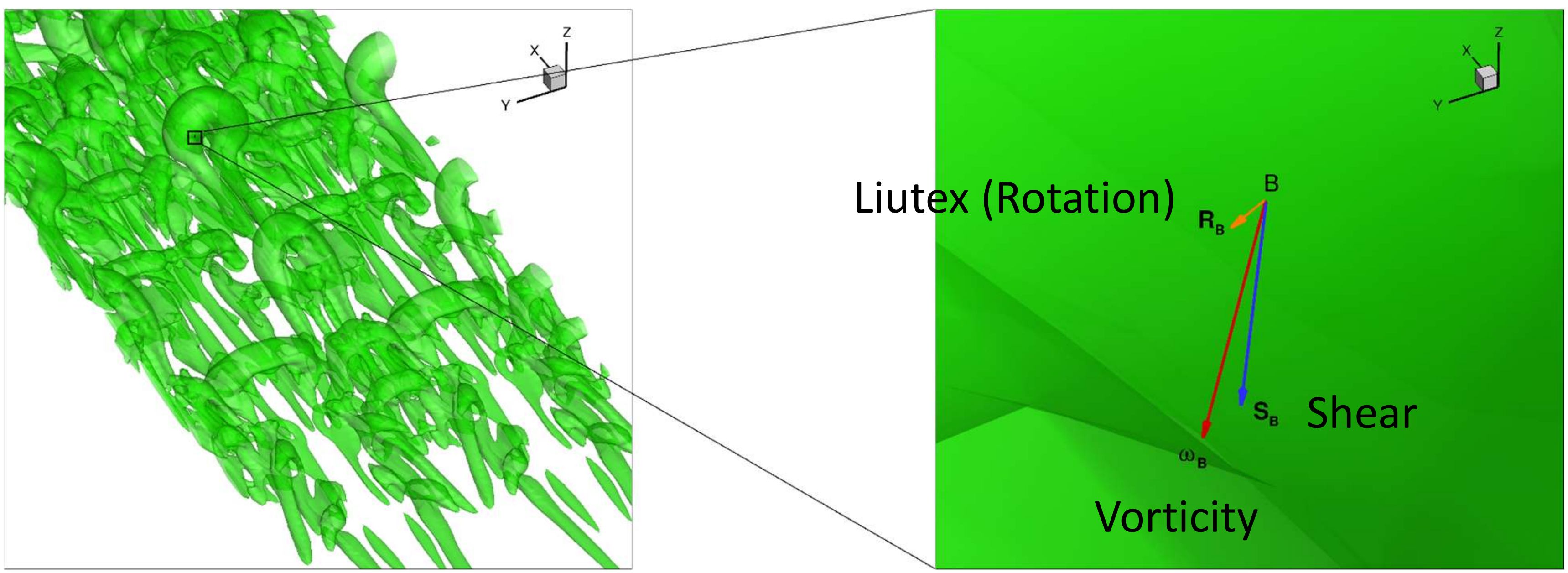


Fig. 4 Illustration of vorticity vector decomposition of Point B.

Vorticity mainly represents shear with small part as vortex/Liutex

New *UTA R-NR* Tensor Decomposition

$$\nabla\vec{\boldsymbol{V}} = \begin{bmatrix} \lambda_{cr} & \frac{R}{2}+\epsilon & \xi \\ -\frac{R}{2} & \lambda_{cr} & \eta \\ 0 & 0 & \lambda_r \end{bmatrix} = \boldsymbol{R} + \boldsymbol{NR} \text{ in principal coordinate}$$

$$\boldsymbol{R} = \begin{bmatrix} 0 & R/2 & 0 \\ -R/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ **- Rigid Rotation**

$$\boldsymbol{NR} = \begin{bmatrix} \lambda_{cr} & \epsilon & \xi \\ 0 & \lambda_{cr} & \eta \\ 0 & 0 & \lambda_r \end{bmatrix}$$ **– No Rotation** (more general for both vortex and non-vortex points

$\epsilon = \frac{\partial U}{\partial Y} - \frac{R}{2}$, $\xi = \frac{\partial U}{\partial Z}$, η $= \frac{\partial V}{\partial Y}$, $\lambda_r - real\ eigenvalue, \lambda_{cr}$-real part of the complex eigenvalue

In laminar boundary layer ***R=0***, but maybe $\lambda_1 \neq \lambda_2 \neq \lambda_{cr}$

New *UTA R-NR* Tensor Decomposition in the xyz coordinate system

$$\nabla\vec{\boldsymbol{v}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & R_z & -R_y \\ -R_z & 0 & R_x \\ R_y & -R_x & 0 \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} - \frac{1}{2}R_z & \frac{\partial w}{\partial x} + \frac{1}{2}R_y \\ \frac{\partial u}{\partial y} + \frac{1}{2}R_z & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} - \frac{1}{2}R_x \\ \frac{\partial u}{\partial z} - \frac{1}{2}R_y & \frac{\partial v}{\partial z} + \frac{1}{2}R_x & \frac{\partial w}{\partial z} \end{bmatrix}$$

$$= \boldsymbol{R} + \boldsymbol{NR}$$

$$(\nabla\vec{\boldsymbol{v}})^{\boldsymbol{T}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -R_z & R_y \\ R_z & 0 & -R_x \\ -R_y & R_x & 0 \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{1}{2}R_z & \frac{\partial u}{\partial z} - \frac{1}{2}R_y \\ \frac{\partial v}{\partial x} - \frac{1}{2}R_z & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{1}{2}R_x \\ \frac{\partial w}{\partial x} + \frac{1}{2}R_y & \frac{\partial w}{\partial y} - \frac{1}{2}R_x & \frac{\partial w}{\partial z} \end{bmatrix} = \boldsymbol{R}^{\boldsymbol{T}} + (\boldsymbol{NR})^{\boldsymbol{T}}$$

NR part satisfies Stokes assumption, not R-part which cannot be isotropic

4. Divergence of a Tensor

Divergence of velocity gradient

$$[\nabla\cdot(\nabla\vec{v})^T]^T = \left(\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{bmatrix}\right)^{\boldsymbol{T}} = \begin{bmatrix}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x}\\ \frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}\\ \frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\\ \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\end{bmatrix}$$

$$[\nabla\cdot(\nabla\vec{v})^T]^T = \begin{bmatrix}\frac{\partial}{\partial x}tr\\ \frac{\partial}{\partial y}tr\\ \frac{\partial}{\partial z}tr\end{bmatrix} = \nabla tr = \nabla(\nabla\cdot\vec{v})$$

For incompressible flow: $[\nabla\cdot(\nabla\vec{v})^T]^T \equiv 0$

Useful for understanding fluid dynamics governing equations:

For incompressible flow:

$\nabla\cdot\nabla\vec{v} = \nabla\cdot\nabla\vec{v} + \nabla\cdot(\nabla\vec{v})^T = \nabla\cdot[\nabla\vec{v} + (\nabla\vec{v})^T]$ - Strain (symmetric)

$\nabla\cdot\nabla\vec{v} = \nabla\cdot\nabla\vec{v} - \nabla\cdot(\nabla\vec{v})^T = \nabla\cdot[\nabla\vec{v} - (\nabla\vec{v})^T]$ - Vorticity (anti-symmetric)

Same to use strain (6 entries in N-S) or vorticity (3 entries in my new governing equations)

5. Strain-Based Navier-Stokes Equations

The original Navier-Stokes equations can be written as

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v}) = \rho\vec{f} - \nabla p - \frac{2}{3}\nabla[\mu(\nabla.\vec{v})] + \{\nabla\cdot[\mu(\nabla\vec{v} + (\nabla\vec{v})^T)]\}^T,$$

1. NS does not have vorticity terms which are important for turbulence research
2. Strain and stretching terms (diagonal and off-diagonal terms) are not Galilean invariant and strongly dependent on coordinates
3. Physical meaning of diagonal and off-diagonal elements are not clear

Since $\nabla\cdot(\nabla\vec{v})^T = \nabla(\nabla\cdot\vec{v})$, the new governing equation is

$$\boldsymbol{\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v}) = \rho\vec{f} - \nabla p + \frac{4}{3}\nabla[\mu(\nabla.\vec{v})] + \{\nabla\cdot[\mu(\nabla\vec{v} - (\nabla\vec{v})^T)]\}^T}$$

1. It has vorticity terms only (no symmetric strain)
2. It only has three anti-symmetric off-diagonal elements (computation is half)
3. The physical meaning of off-diagonal elements are anti-symmetric shear

6. Vorticity-Based New Governing Equations

Mathematical Foundation

Theorem 1: $\nabla \cdot (\nabla \vec{v})^T = \nabla(\nabla \cdot \vec{v})$ - Easy to prove (see Liu & Liu 2021)

Corollary 1: Velocity gradient, strain, and vorticity are transferable.
$\nabla \cdot [\mu(\nabla \vec{v} + (\nabla \vec{v})^T)] = \nabla \cdot [\mu(\nabla \vec{v} - (\nabla \vec{v})^T)] + \nabla 2\mu(\nabla \cdot \vec{v}) = \nabla \cdot \mu \nabla \vec{v} + \nabla(\nabla \cdot \vec{v})$

For incompressible flow: $\nabla \cdot \vec{v} = 0$, $\nabla \cdot (\nabla \vec{v})^T = 0$

Corollary 2: $\nabla \cdot [\mu(\nabla \vec{v} + (\nabla \vec{v})^T)] = \nabla \cdot [\mu(\nabla \vec{v} - (\nabla \vec{v})^T)] = \nabla \cdot \mu \nabla \vec{v}$

Symmetric strain and anti-symmetric vorticity tensors are equivalent for incompressible flow, but need one divergence term for interchange in compressible flow

6. Vorticity-Based New Governing Equations

The original Navier-Stokes equations assume:

1. Strain is symmetric
2. Stress is proportional to strain (Stokes assumption)
3. Both strain and stress are symmetric
4. There is no role of vorticity

The new governing equations assume

1. Stress is proportional to vorticity
2. Both vorticity and stress are anti-symmetric
3. There is no role of symmetric strain. (actually both strain and vorticity have roles)

As shown below, they are equivalent.

6. Vorticity-Based New Governing Equations

The viscous terms are changed from symmetric to anti-symmetric and six elements become three elements:

$$(\nabla\vec{v} - (\nabla\vec{v})^T) = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}$$

The stresses in the original NS equation is

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k},\ \ i, j, k = 1, 2,3,$$

with six independent elements because it is symmetric

In the new governing equation,

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \frac{4}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}, \qquad i, j, k = 1,2,3, i \neq j$$

with three independent elements because it is anti-symmetric

6. Vorticity-Based New Governing Equations

For incompressible flow,

$\nabla \cdot (\nabla \vec{v})^T = \nabla(\nabla \cdot \vec{v}) = 0,$

The original NS equation is

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla \cdot (\rho\vec{v}\vec{v}) = \rho\vec{f} - \nabla p + \{\nabla \cdot [\mu(\nabla\vec{v} + (\nabla\vec{v})^T)]\}^T$$

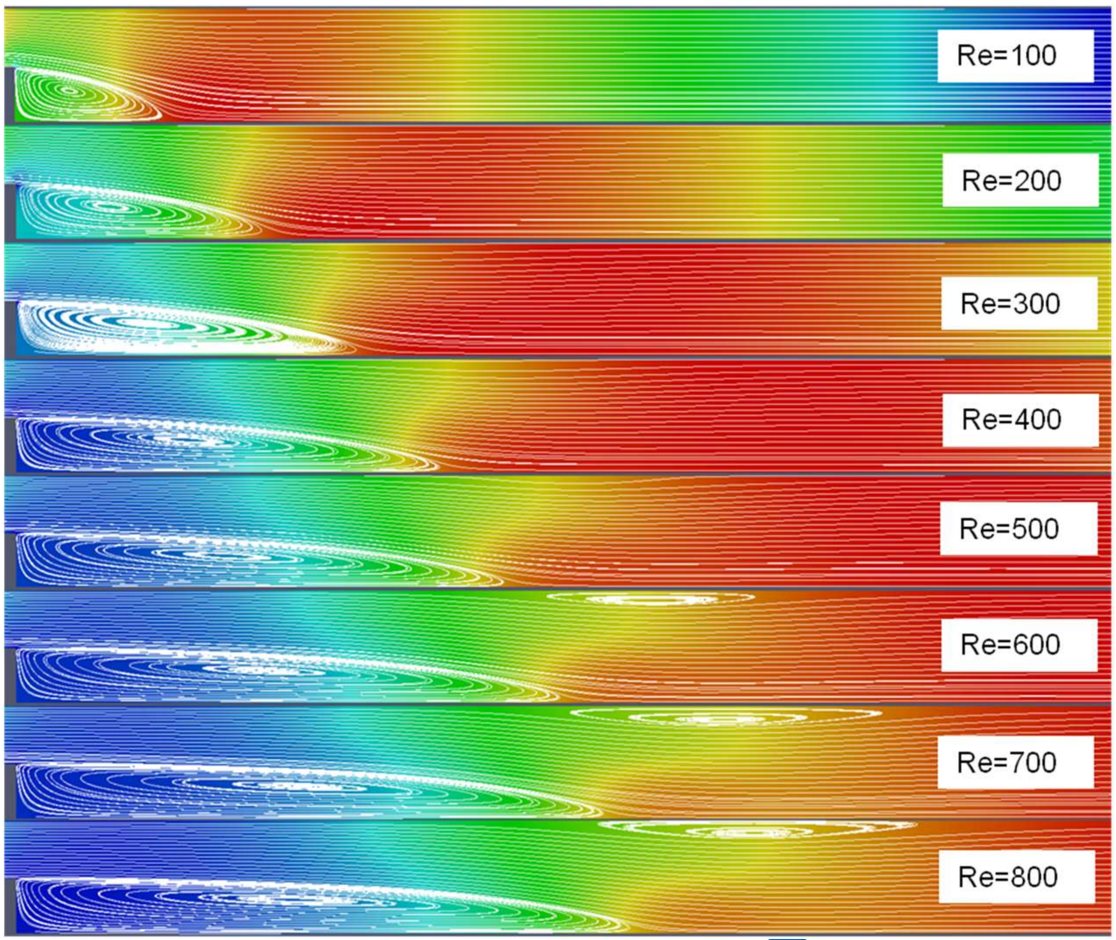
The new governing equation is

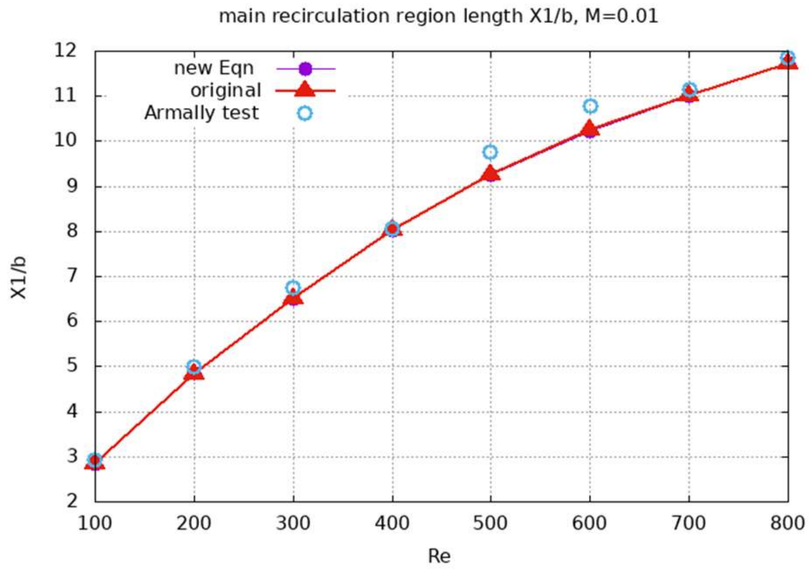
$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla \cdot (\rho\vec{v}\vec{v}) = \rho\vec{f} - \nabla p + \{\nabla \cdot [\mu(\nabla\vec{v} - (\nabla\vec{v})^T)]\}^T$$

Note that $\nabla\vec{v}$ has 9 terms, $\nabla\vec{v} + (\nabla\vec{v})^T$ has 6 terms, $\nabla\vec{v} - (\nabla\vec{v})^T$has 3 terms only (save half)

6. New Governing Equations –Test Cases

Computational Results (2-D back step laminar flow)

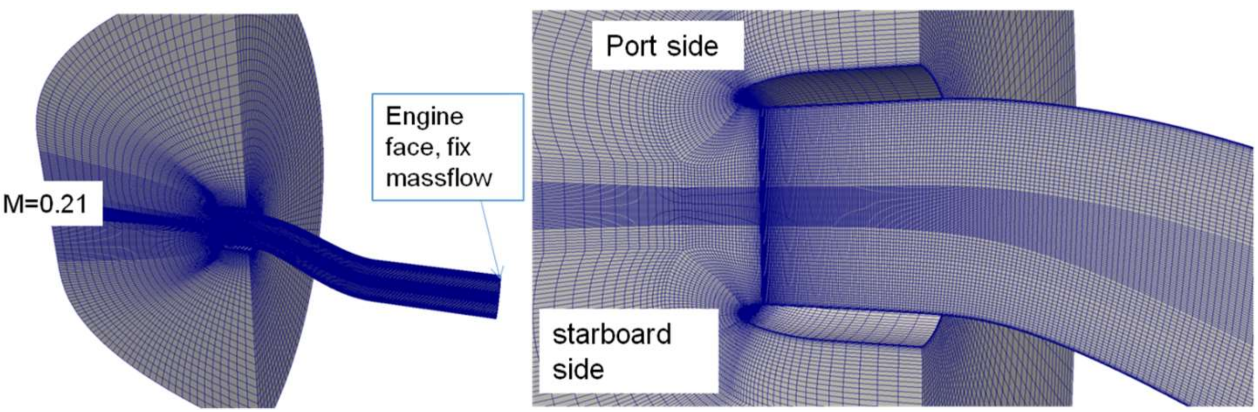


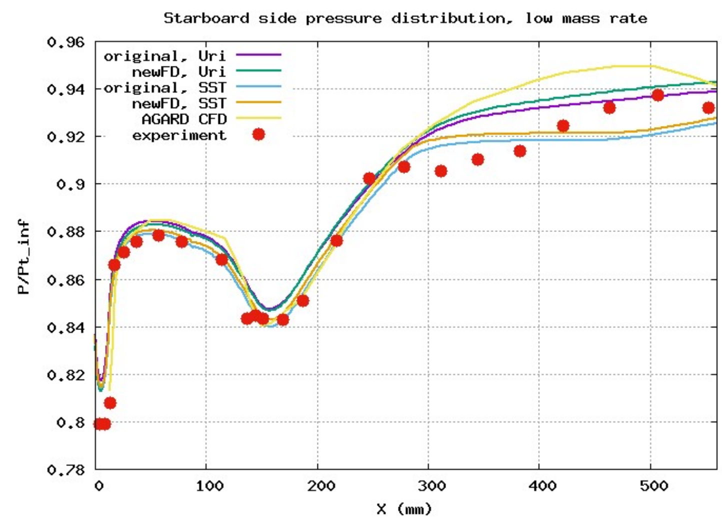


The computational results by the new governing equations are almost same as by Navier-Stokes.

6. New Governing Equations – Test Cases

Computational Results – Turbulent Flow in an S-Duct





The computational results by the new governing equations are almost same as by Navier-Stokes.

Some ideas on governing equations of fluid dynamics

As $\nabla\vec{v} = \widetilde{R} + \widetilde{S} + \widetilde{SC}$, we should consider forces produced by rotation and stretch

$$\tilde{F} = QFQ^T = Q\left\{\mu_1\begin{bmatrix} -R^2/4 & 0 & 0 \\ 0 & -R^2/4 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (\mu_2-\mu_4)\begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_r \end{bmatrix} + \mu_4\frac{1}{2}\left(\nabla V + \nabla V^T\right)\right\}Q^T \quad (19)$$

$$\tilde{F} = \mu_1\left\{\begin{bmatrix} -R^2/4 & 0 & 0 \\ 0 & -R^2/4 & 0 \\ 0 & 0 & -R^2/4 \end{bmatrix} + \frac{R^2}{4}\begin{bmatrix} r_x^2 & r_x r_y & r_x r_z \\ r_y r_x & r_y^2 & r_y r_z \\ r_z r_x & r_z r_y & r_z^2 \end{bmatrix}\right\} + (\mu_2-\mu_4)\left\{\begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_{cr} \end{bmatrix} + (\lambda_r - \lambda_{cr})\begin{bmatrix} r_x^2 & r_x r_y & r_x r_z \\ r_y r_x & r_y^2 & r_y r_z \\ r_z r_x & r_z r_y & r_z^2 \end{bmatrix}\right\} + \mu_4\frac{1}{2}\left(\nabla v + \nabla v^T\right) - \tilde{p}I$$

Note that $\boldsymbol{\mu_4 = \mu}$ *here*

New governing equations for fluid dynamics at vortex points (non-rotational points are similar)

$$\rho\frac{dv}{dt} = \nabla\cdot F + \rho f$$

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}\boldsymbol{v}) = \rho\boldsymbol{f} - \nabla p - \frac{2}{3}\nabla[\mu(\nabla.\boldsymbol{v})] + \nabla\cdot[\mu(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)] + \nabla\cdot(\widetilde{\boldsymbol{F}_1} + \widetilde{\boldsymbol{F}_2}) \quad (20)$$

$$\tilde{F}_1 = \mu_1\left\{\begin{bmatrix} -R^2/4 & 0 & 0 \\ 0 & -R^2/4 & 0 \\ 0 & 0 & -R^2/4 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} R_x^2 & R_x R_y & R_x R_z \\ R_y R_x & R_y^2 & R_y R_z \\ R_z R_x & R_z R_y & R_z^2 \end{bmatrix}\right\}$$

$$\tilde{F}_2 = (\mu_2-\mu_4)\left\{\begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_{cr} \end{bmatrix} + \frac{(\lambda_r - \lambda_{cr})}{R^2}\begin{bmatrix} R_x^2 & R_x R_y & R_x R_z \\ R_y R_x & R_y^2 & R_y R_z \\ R_z R_x & R_z R_y & R_z^2 \end{bmatrix}\right\}$$

$\widetilde{\boldsymbol{F}_1}$ *is produced by rotation and* $\widetilde{\boldsymbol{F}_2}$ is produced by stretch

C. Liu, New ideas on governing equations of fluid dynamics. *J Hydrodyn* 33, 861–866 (2021); https://doi.org/10.1007/s42241-021-0050-8

Conclusions

1. Velocity gradient tensor is unique, but corresponding matrix is countless.
2. We need a unique coordinate system (Principal Coordinate System) to get a unique Principal Matrix
3. Principal decomposition is unique $\nabla\vec{V} = R + SC + S$ =Rotation+Stretch+Shear)
4. Principal decomposition can be done in the original Cartesian coordinates to establish the new fluid kinematics
5. Stress calculated by velocity gradient, strain or vorticity is equivalent
6. Velocity gradient has 9 elements, strain has 6, but vorticity only has 3 (anti-symmetric)
7. The new governing equation by vorticity is Galilean invariant, simpler, and has clear physical meaning.
8. The new governing equation has vorticity (NS does not have) which can be further decomposed to rigid rotation and pure anti-symmetric shear. These new ideas may be useful for turbulence research.

Recent Publications

1. C. Liu and Z. Liu, New Governing Equations for Fluid Dynamics, *AIP Advances* 11, 115025 (2021); https://doi.org/10.1063/5.0074615
2. C. Liu, New ideas on governing equations of fluid dynamics. *J Hydrodyn* 33, 861–866 (2021); https://doi.org/10.1007/s42241-021-0050-8
3. C. Liu, New fluid kinematics, *Journal of Hydrodynamics*, 2021, https://doi.org/10.1007/s42241-021-0037-5.
4. C. Liu, Y. Gao, X. Dong, J. Liu, Y. Zhang, X. Cai, N. Gui, "Third generation of vortex identification methods: Omega and Liutex/Rortex based systems", *Journal of Hydrodynamics* (2019), 31(2): 1-19,
5. C. Liu, Y. Gao, S. Tian, X. Dong, "Rortex—A new vortex vector definition and vorticity tensor and vector decompositions" *Physics of Fluids*, 30, 035103 (2018); doi: 10.1063/1.5023001.
6. Gao, Y. and Liu, C., "Rortex and comparison with eigenvalue-based identification criteria", *Physics of Fluid* **30**, 085107 (2018).
7. Y. Wang, Y. Gao, J. Liu, C. Liu, "Explicit formula for the Liutex vector and physical meaning of vorticity based on the Liutex-Shear decomposition", *J Hydrodyn* (2019). https://doi.org/10.1007/s42241-019-0032-2 with Y. Wang, Y. Gao, Y. Liu

Thank you!