UNIVERSITY OF TEXAS ARLINGTON

Mathematics Foundation of Liutex

Lecture 2 of Liutex Short Course

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Presented for

Online Liutex Short Course

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Outlines

I. My Confusions
II. Vector, Tensor and Matrix
III. Tensor Operation and Matrix Operation
IV. Hamilton Operator
V. Velocity Gradient Tensor and Matrix
VI. Rotation of Coordinate System
VII. Liutex Definition
VIII. Divergence of the Tensor
IX. Conclusions

1. Vortex is Ubiquitous in Universe

(a) Tornado

(b) Hurricane

(c) Airplane tip vortex

(d) Galaxy

1.1 My Confusions with textbooks

1.Helmholtz's original view

Vorticity Line → Vorticity Filaments → Vorticity Tube → Vortex

2. Vorticity: A clear mathematic definition, namely the curl of the velocity vector $\vec{v}$: $\vec{\omega} \equiv \nabla \times \vec{v}$

3. In most fluid dynamics textbooks: first says vortex is vorticity tube and late says "turbulence is generated by vortex breakdown".

4. My confusions:

1) $\nabla \cdot (\nabla \times \vec{v}) \equiv 0$, which means vortex (vorticity tube) can never break down (Liu et al. 2014)

2) Turbulence is generated by vortex breakdown which can never happen

- This is a serious contradiction in textbooks – later part against the early part in the same textbook

Vortex is a natural phenomenon, but vorticity is a mathematical definition. Vortex=vorticity?

H. Helmholtz, "Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen," Journal für die reine und angewandte Mathematik 55, 25-55 (1858).

1.2 Some Examples by Textbooks

MIT Online Lecture Notes on Fluids (2008) https://web.mit.edu/16.unified/www/SPRING/fluids/Spring2008/LectureNotes/f06.pdf

Fluids – Lecture 6 Notes

1. 3-D Vortex Filaments
2. Lifting-Line Theory

Reading: Anderson 5.1

3-D Vortex Filaments

General 3-D vortex

A 2-D vortex, which we have examined previously, can be considered as a 3-D vortex which is straight and extending to $\pm\infty$. Its velocity field is

$$V_\theta = \frac{\Gamma}{2\pi r} \qquad V_r = 0 \qquad V_z = 0 \qquad \text{(2-D vortex)}$$

In contrast, a general 3-D vortex can take any arbitrary shape. However, it is subject to the *Helmholtz Vortex Theorems*:

1) The strength Γ of the vortex is constant all along its length
2) The vortex cannot end inside the fluid. It must either
   a) extend to $\pm\infty$, or
   b) end at a solid boundary, or
   c) form a closed loop.

Proofs of these theorems are beyond scope here. However, they are easy to apply in flow modeling situations.

Apparently, they think vortex is a vorticity tube. *However, 1) vorticity line cannot end on solid wall (u=v=0, $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$), 2) Helmholtz three theorems only work for inviscid flow, but turbulence cannot be inviscid,* *3) circulation* Γ is not fluid rotation (e.g. lminar channel)

1.2 Some Examples by Textbooks

- Wu et al.: vortex is "**a connected fluid region with high concentration of vorticity compared with its surrounding.**"

- Nitsche (Encyclopedia): **A vortex is commonly associated with the rotational motion of fluid around a common centerline. It is defined by the vorticity in the fluid, which measures the rate of local fluid rotation.**
  – That is incorrect

H. Lamb, *Hydrodynamics*, (Cambridge university press, Cambridge, 1932).
P. Saffman, *Vortices dynamics*, (Cambridge university press, Cambridge, 1992).
J.-Z. Wu, H.-Y. Ma, and M.-D. Zhou, *Vorticity and vortices dynamics*, (Springer-Verlag, Berlin Heidelberg, 2006).
M. Nitsche, "Vortex Dynamics," in Encyclopedia of Mathematics and Physics, (Academic Press, New York, 2006).

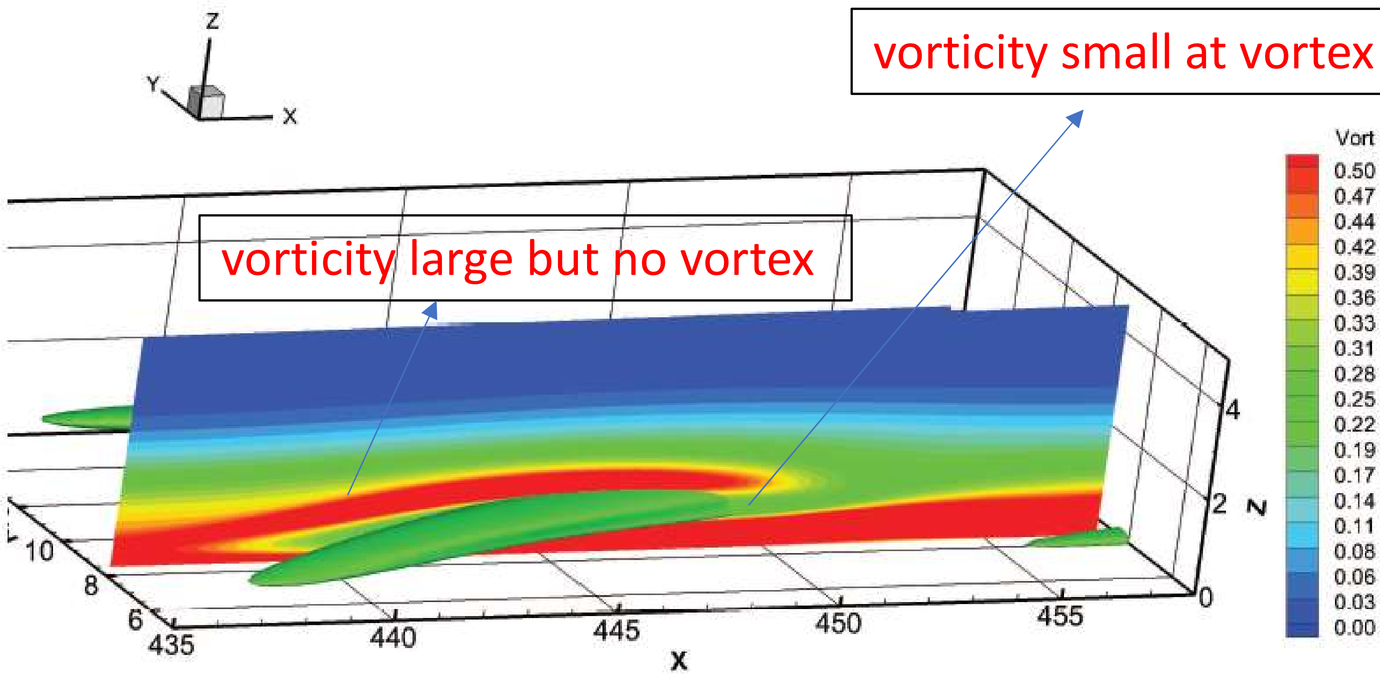
1.3 Vorticity-based definitions and limitations (vortex cannot be identified by vorticity)

- Vortex is a natural phenomenon, but vorticity is a mathematical definition. How do we know vortex is vorticity?
- **Immediate counter-example is the laminar boundary layer where the vorticity (shear) is very large, but not rotation (no vortex) exists, which will lead to the conclusion : vortex cannot be described by vorticity**
- **Same thing happens in a laminar channel flow**

1.3 Vorticity-based definitions and limitations

- The maximum vorticity does not necessarily occur in the central region of vortical structures. As pointed out by Robinson(1989), **"the association between regions of strong vorticity and actual vortices can be rather weak in the turbulent boundary layer, especially in the near wall region."** Wang et al.(Communication in Computational Physics, 2016) obtain a similar result that the magnitude of vorticity inside a Lambda vortex can be substantially smaller than the surrounding near the solid wall in a flat plate boundary layer.

1.3 Vorticity-based definitions and limitations (vortex cannot be identified by vorticity)

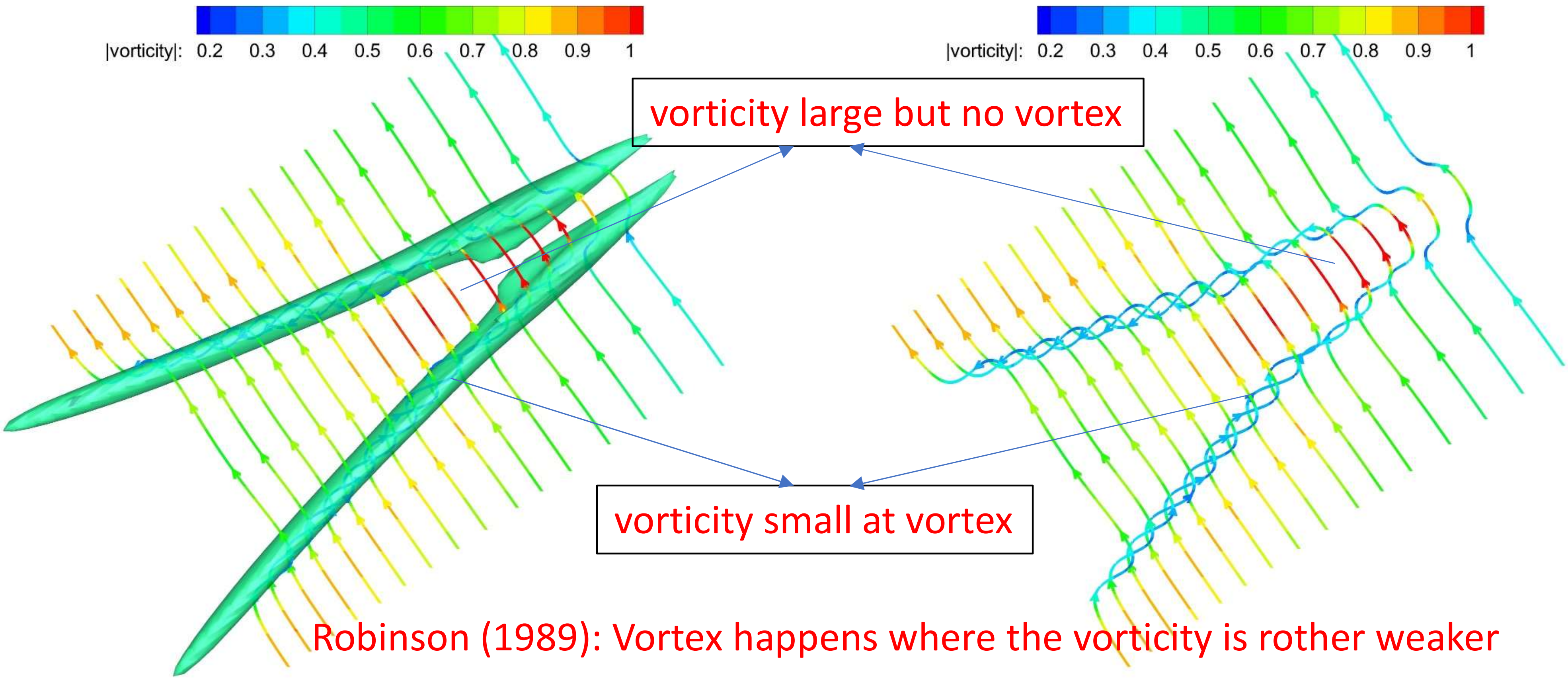


Λ vortex and shear layer

Y. Wang, Y. Yang, G. Yang and C. Liu, “DNS study on vortex and vorticity in late boundary layer transition,” Comm. Comp. Phys. **22**, 441-459 (2017).

Robinson (1989): Vortex happens where the vorticity is rother weaker

1.3 Vorticity-based definitions and limitations (vortex cannot be identified by vorticity)



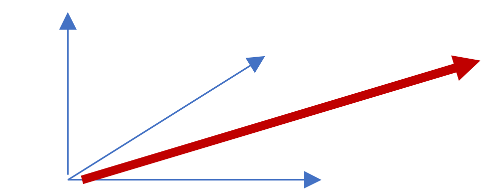
Robinson (1989): Vortex happens where the vorticity is rother weaker

C. Liu, Y. Gao, S. Tian, and X. Dong, "Rortex—A new vortex vector definition and vorticity tensor and vector decompositions," Phys. Fluids 30, 035103 (2018).

2. Vector, Tensor and Matrix

1) Difference between Vector/Tensor and Matrix

	Vector/Tensor	Matrix
Meaning	Physics	Mathematics
Features	Objective/Galilean Invariant	Dependent on Coordinates
	Unique	Infinity
Operations	Dot, Cross, Dyadic	Plus, Subtraction, Multiplication,
		Inversion, Transpose



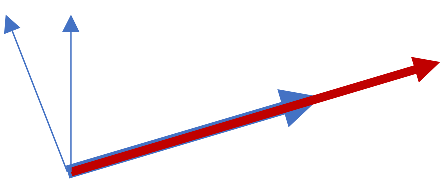
$$\vec{V} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

1) Do not think matrix is unique for vector/tensor

$e.g.$ $\vec{V}$ has infinity number of corresponding 3x1 matrices

2) Vector/tensor **dot product is not matrix multiplication**

3) $\nabla\vec{V} \neq (\nabla\vec{V})^T$ if $\nabla\vec{V}$ is not symmetric



$$\vec{V} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

3. Vector, Tensor and Matrix Operations

Note that vector/tensor **dot product is not matrix multiplication (cannot drop the dot)**

$\nabla\vec{V} \neq (\nabla\vec{V})^T$ if $\nabla\vec{V}$ is not symmetric

1. Vector Dot Product

$$\vec{a}\cdot\vec{b} = \vec{a}^T\vec{b} = [a_1\ \ a_2\ \ a_3\ ]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \sum_1^3 a_i\, b_i$$

Vector dot product   Matrix multiplication ($[a_1\ \ a_2\ \ a_3\ ] = \vec{a}^T$)

2. Tensor/Vector Dot Product

Tensor dot product

$$\boldsymbol{A}\cdot\vec{b} = \boldsymbol{A}^T\vec{b} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{\boldsymbol{T}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^3 a_{i1}\, b_i \\ \sum_{i=1}^3 a_{i2}\, b_i \\ \sum_{i=1}^3 a_{i3}\, b_i \end{bmatrix}$$

Matrix multiplication (transpose)

Note that $\boldsymbol{A}$ becomes $\boldsymbol{A}^T$

3. Tensor/Vector Dyadic Product $\vec{a}\otimes\vec{b} = \vec{a}\,(\vec{b})^T$

All vector/tensor and operations should be transfer to matrix and matrix operations

4. Hamilton operators:

1. Hamilton operator: $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$

2. $\nabla \cdot \vec{v} \quad = \quad \nabla^T \vec{v} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

vector divergence   matrix multiplication (left transpose is required)

3. $\nabla \vec{v} = \nabla \otimes \vec{v} = \nabla(\vec{v}^T) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u & v & w \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}$

vector dyadic   matrix multiplication (right transpose is required)

We should not simply drop $\otimes$ which is a dyadic operator, and must do transpose $\vec{v}$ before drop $\otimes$

Eigenvalue and Eigenvector

Definition: Let A be an nxn matrix. Any values of λ such that

$A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = 0$

has nonzero solutions λ are called eigenvalues of A. The corresponding nonzero vectors $\vec{v}$ are called eigenvectors of A. *Example*: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$p(\lambda) = det\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = -\lambda^3 + 1 + 1 + \lambda + \lambda + \lambda = -\lambda^3 + 3\lambda + 2 = 0$$

$-\lambda^3 + 3\lambda + 2 = (\lambda + 1)(-\lambda^2 + \lambda + 2) = -(\lambda + 1)(\lambda + 1)(\lambda - 2) = 0, \;\; \lambda = -1$ (double roots) and 2

Eigenvectors: $E_{-1} = Span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$, $E_2 = Span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$, Eigenvecotor is not unique but a sub-space.

Eigenvalue and Eigenvector

Example: $A = \begin{bmatrix} 3 & -2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$p(\lambda) = det\begin{bmatrix} 3-\lambda & -2 & 0 \\ 4 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (3-\lambda)(-1-\lambda)(1-\lambda) - (1-\lambda)(-2)(-4) = -\lambda^3 + 3\lambda^2 - 7\lambda + 5$$

$= (\lambda - 1)\ (\lambda^2 - 2\lambda + 5)$=0

The eigenvalues of A are 1, 1-2i and 1+2i (two conjugate complex eigenvalues).

There must be one real eigenvector:

$$\begin{bmatrix} 3-1 & -2 & 0 \\ 4 & -1-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \rightarrow \text{Gaussian} \rightarrow \begin{bmatrix} 2 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow x = 0, y = x = 0, z = 1 \text{ (or any k)}$$

One real Eigenvector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as $\begin{bmatrix} 3 & -2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

In the Vortex Area: $(\nabla\vec{v})^T$ or $\nabla\vec{v}$ has one real eigenvalue (one real eigenvector) and two conjugate complex eigenvalues

5. Velocity Gradient Tensor and 3x3 Matrix

Velocity increment
According to physical definition:

$$d\vec{v} = \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \\ \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \\ \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \nabla\vec{v} \cdot d\vec{l}$$

left transpose is required by dot product

which is exactly the physical definition that the increment of velocity on a line $d\vec{l}$ is the velocity gradient projection on the line. Therefore, $\nabla\vec{v}$ is the velocity gradient tensor.

4. Velocity Gradient Tensor and Matrix

Misunderstanding made by Wiki and most western fluid dynamics textbooks

Wiki has definition on velocity gradient tensor (see https://en.wikipedia.org/wiki/Strain-rate_tensor;)

It says that "in continuum mechanics, in 3-dimensions, the gradient of the velocity $\boldsymbol{\nabla}\vec{\boldsymbol{v}}$ is a second-order tensor J (see below) which can be transposed as the matrix L"

$$L=(\boldsymbol{\nabla}\vec{\boldsymbol{v}})^{\boldsymbol{T}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \text{ or } \boldsymbol{\nabla}\vec{\boldsymbol{v}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

which is incorrect!

However, the right definition is

$$\nabla\vec{v} = \nabla\otimes\vec{v} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \neq (\boldsymbol{\nabla}\vec{\boldsymbol{v}})^{\boldsymbol{T}}$$

Apparently, in Wiki and most western fluid dynamics textbooks, the gradient tensor of velocity $\boldsymbol{\nabla}\vec{\boldsymbol{v}}$ is really defined as $(\nabla\vec{v})^T$. Many people think it is ok to treat a matrix and the transpose of a matrix as identical. However, transpose matrix has same eigenvalues but different eigenvectors and will cause serious mistakes in research on fluid dynamics.

5. Liutex Definition

1. Velocity Gradient

$$\nabla\vec{v} = \nabla \otimes \vec{\boldsymbol{v}}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [u \;\; v \;\; w] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}$$

Right!

***in wiki and most western fluid dynamics textbooks*:**

$$\nabla\vec{v} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$ is Incorrect!!

Confused by dyadic vectors and matrix multiplication

https://en.wikipedia.org/wiki/Strain-rate_tensor

$\nabla\vec{v}$ here implicates $\nabla \otimes \vec{\boldsymbol{v}}$

2. Velocity increment

$$d\vec{v} = \nabla\vec{v} \cdot d\vec{l} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \bullet \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = (\nabla\vec{v})^T \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \\ \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \\ \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \end{bmatrix}$$

Dot product

Transposed for Matrix multiplication

5. Liutex Definition

Liutex

$$\nabla\vec{v}\cdot\vec{r}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}\end{bmatrix}\bullet\begin{bmatrix}r_x\\ r_y\\ r_z\end{bmatrix}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{bmatrix}\begin{bmatrix}r_x\\ r_y\\ r_z\end{bmatrix}=\lambda\vec{r}$$

Tensor dot product Matrix multiplication – Transpose is required

$d\vec{v}=\nabla\vec{v}\cdot\vec{r}=\lambda\vec{r}$ - Stretch only – that is Liutex vector!

Liutex is eigenvector of matrix $(\nabla\vec{v})^T$ Not $\nabla\vec{v}$ which is the reason why people spent so long time (160 years) to find Liutex.

Tensor does not have eigenvector, but matrix has!

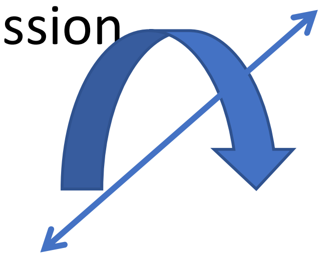
5. Liutex Definition

What is the Local Rotation Axis?

Definition 1: A local fluid rotation axis is defined as a vector that can only have stretching (compression) along its length.

$$d\vec{v} = \boxed{\underbrace{\nabla\vec{v}\cdot\vec{r}}_{\text{Dot product}}} = \boxed{\underbrace{\nabla\vec{v}^{T}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \alpha\,\vec{r}}_{\text{Matrix multiplication}}}$$

- stretching or compression



- $\vec{r}$ is eigenvector of matrix $(\nabla\vec{v})^T$ but not $\nabla\vec{v}$

Here we limited $\vec{r}$ by the condition of $\vec{\omega}\cdot\vec{r} > 0 \; and \; \|\vec{r}\|_2$=1

What is the eigenvector of $\nabla\vec{v}$ $\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$ - $\vec{t} \neq \vec{r}$ is not Liutex

Only matrix has eigenvector, but tensor has projection or dot product

5. Liutex Definition

1. Transpose is for matrix not for tensor

2. Dot product is commutable - answer to the left eigenvector question

$$\nabla\vec{v}\cdot\vec{r} = (\vec{r}\cdot\nabla\vec{v})^T = \lambda\vec{r}$$

$$\text{left side} = \nabla\vec{v}\cdot\vec{r} = (\nabla\vec{v}\,)^T\,\vec{r} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} r_x + & \frac{\partial u}{\partial y} r_y + & \frac{\partial u}{\partial z} r_z \\ \frac{\partial v}{\partial x} r_x + & \frac{\partial v}{\partial y} r_y + & \frac{\partial v}{\partial z} r_z \\ \frac{\partial w}{\partial x} r_x + & \frac{\partial w}{\partial y} r_y + & \frac{\partial w}{\partial z} r_z \end{bmatrix}$$

$$\text{right side} = \vec{r}\cdot\nabla\vec{v} = \vec{r}^T\nabla\vec{v} = \begin{bmatrix} r_x & r_y & r_z \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} r_x + & \frac{\partial u}{\partial y} r_y + & \frac{\partial u}{\partial z} r_z \\ \frac{\partial v}{\partial x} r_x + & \frac{\partial v}{\partial y} r_y + & \frac{\partial v}{\partial z} r_z \\ \frac{\partial w}{\partial x} r_x + & \frac{\partial w}{\partial y} r_y + & \frac{\partial w}{\partial z} r_z \end{bmatrix}^T$$

$(\vec{r}^T\nabla\vec{v})^T = (\nabla\vec{v}\,)^T\,\vec{r}$ Left eigenvector needs transpose of $\nabla\vec{v}$ matrix

Only matrix has eigenvector, but tensor has projection or dot product

5. Liutex Definition

$\nabla\vec{v}^T \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \alpha\,\vec{r}$ There is only stretching (compression) along $\vec{r}$ -local rotation axis

- $\vec{r}$ is eigenvector of matrix $(\nabla\vec{v}\,)^T$ but not $\nabla\vec{v}$ and

$$\vec{R} = R\vec{r} = \left\{\langle\vec{\omega},\vec{r}\rangle - \sqrt{\langle\vec{\omega},\vec{r}\rangle^2 - 4\lambda_{ci}^2}\right\}\vec{r}$$

$\vec{r}$ is the Liutex direction and, therefore, answers why taking so long time to find Liutex which is the rigid mathematical definition of local rotation or vortex

Rotation of Coordinate System

2-D Coordinate System Rotation:

$\begin{bmatrix} x \\ y \end{bmatrix}$ in the original coordinates, the coordinates $\begin{bmatrix} X \\ Y \end{bmatrix}$ in the new (rotated) coordinates can be expressed as $\begin{bmatrix} X \\ Y \end{bmatrix} = \boldsymbol{P} \begin{bmatrix} x \\ y \end{bmatrix}$

$\boldsymbol{P} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$ is a rotation matrix which is orthogonal, namely, $\boldsymbol{P}\boldsymbol{P}^T = \boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{I}$.

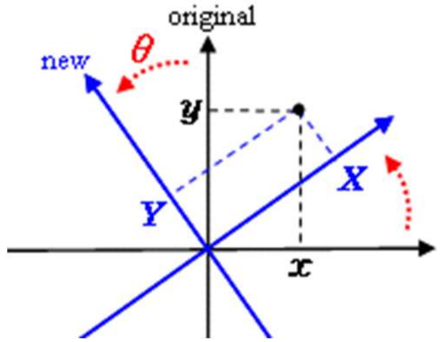


Figure 2.1 2D Coordinate rotation

Velocity gradient tensor in the rotated coordinate system

If a rotation matrix P is used to rotate the *xy*-frame to *XY*-frame, the velocity gradient tensor in the *XY*-frame $\nabla\vec{V}$ is related to the velocity gradient tensor in the *xy*-frame $\nabla\vec{v}$ through the following expression:

$$(\nabla\vec{V})^T = P^{-1}(\nabla\vec{v})^T P = P^T(\nabla\vec{v})^T P; \;\; \nabla\vec{V} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{bmatrix}$$

.

Rotation of Coordinate System

Principal Rotation and Principal Coordinates in 3D

Rotation matrix $\boldsymbol{Q}$ in the xyz coordinate system

Definition 3. $\boldsymbol{Q}^T$ is defined as a rotation matrix to rotate the z-axis to parallel to $\vec{r}$,

where $\boldsymbol{Q} = [Q_1 \quad Q_2 \quad Q_3] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$, $\quad Q_1 \perp Q_2, Q_1 \perp Q_3, Q_2 \perp Q_3$

Theorem 1. If the third column of rotation $\boldsymbol{Q}$ is $\vec{r}$, $\boldsymbol{Q}^T$ can rotate the z-axis to parallel to $\vec{r}$**.**

$$\boldsymbol{Q} = \begin{bmatrix} Q_{11} & Q_{12} & r_x \\ Q_{21} & Q_{22} & r_y \\ Q_{31} & Q_{32} & r_z \end{bmatrix}, \boldsymbol{Q}^T \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ r_x & r_y & r_z \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(\nabla\vec{V})^T = Q^{-1}(\nabla\vec{v})^T Q = Q^T(\nabla\vec{v})^T Q = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ r_x & r_y & r_z \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & r_x \\ Q_{21} & Q_{22} & r_y \\ Q_{31} & Q_{32} & r_z \end{bmatrix} = \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} & 0 \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} & 0 \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} & \lambda_r \end{bmatrix}$$

Because $(\nabla\vec{v})^T\ \vec{r} = \lambda_r\ \vec{r}$; $\ Q_1, Q_2$ and $\vec{r}$ and orthogonal

Rotation of Coordinate System

Principal Rotation and Principal Coordinates in 3D

Rotation matrix $\boldsymbol{Q}$ in the xyz coordinate system (In fact, we do not need to do Q and P rotation to get Liutex – Gao &Liu PoF 2018)

Definition 3. $\boldsymbol{Q}^T$ is defined as a rotation matrix to rotate the z-axis to parallel to $\vec{r}$,

where $\boldsymbol{Q} = [Q_1 \quad Q_2 \quad Q_3] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}, \quad Q_1 \perp Q_2, Q_1 \perp Q_3, Q_2 \perp Q_3$

Theorem 2. If the third column of rotation $\boldsymbol{Q}$ is $\vec{r}$, at least one $\boldsymbol{Q}$ can be given by

$$\boldsymbol{Q} = \begin{bmatrix} Q_{11} & Q_{12} & r_x \\ Q_{21} & Q_{22} & r_y \\ Q_{31} & Q_{32} & r_z \end{bmatrix} = \begin{bmatrix} 0 & ar_z & r_x \\ r_z & r_z & r_y \\ -r_y & (-r_y - ar_x) & r_z \end{bmatrix} \text{ where } a = -\frac{{r_y}^2+{r_z}^2}{r_x r_y} \text{ assume } r_x \neq 0 \; and r_y \neq 0$$

Proof: $Q_1 \cdot Q_3 = \begin{bmatrix} 0 \\ r_z \\ -r_y \end{bmatrix} \cdot \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = [0 \quad r_z \quad -r_y] \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = 0$

$Q_2 \cdot Q_3 = \begin{bmatrix} ar_z \\ r_z \\ -r_y - ar_x \end{bmatrix} \cdot \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = [ar_z \quad r_z \quad -r_y - ar_x] \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = ar_z r_x + r_z r_y + (-r_y - ar_x) r_z = ar_z r_x + r_z r_y - r_y r_z - ar_x r_z = 0$

$Q_1 \cdot Q_2 = \begin{bmatrix} 0 \\ r_z \\ -r_y \end{bmatrix} \cdot \begin{bmatrix} ar_z \\ r_z \\ -r_y - ar_x \end{bmatrix} = [ar_z \quad r_z \quad -r_y - ar_x] \begin{bmatrix} 0 \\ r_z \\ -r_y \end{bmatrix} = {r_z}^2 + {r_y}^2 + ar_x r_y = {r_z}^2 + {r_y}^2 - \frac{{r_y}^2+{r_z}^2}{r_x r_y} r_x r_y = 0$

Rotation of Coordinate System

Principal Rotation and Principal Coordinates in 3D :

Rotation matrix *P* **(In fact, we do not need to do Q and P rotation to get Liutex – Gao &Liu PoF 2018)**

Definition 4. $\boldsymbol{P^T}$ is defined as a rotation matrix to rotate the $(\nabla\vec{V})^T$ to the Principal Matrix

i.e. $\boldsymbol{P^T}(\nabla\vec{V})^T P = \left(\nabla\vec{V}_\theta\right)^T = \begin{bmatrix} \lambda_{cr} & -\frac{R}{2} & 0 \\ \frac{R}{2}+\epsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix}$ where $(\nabla\vec{V})^T = \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} & 0 \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} & 0 \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} & \lambda_r \end{bmatrix}$

$\boldsymbol{P^T}$ can be given by $\begin{bmatrix} \cos\theta & -sin\theta & 0 \\ \sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} \cos\theta & -sin\theta & 0 \\ \sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} & 0 \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} & 0 \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} & \lambda_r \end{bmatrix} \begin{bmatrix} \cos\theta & sin\theta & 0 \\ -\sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -sin\theta & 0 \\ \sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial X}\cos\theta - \frac{\partial U}{\partial Y}\sin\theta & \frac{\partial U}{\partial X}\sin\theta + \frac{\partial U}{\partial Y}\cos\theta & 0 \\ \frac{\partial V}{\partial X}\cos\theta - \frac{\partial V}{\partial Y}\sin\theta & \frac{\partial V}{\partial X}\sin\theta + \frac{\partial V}{\partial Y}\cos\theta & 0 \\ \frac{\partial W}{\partial X}\cos\theta - \frac{\partial W}{\partial Y}\sin\theta & \frac{\partial W}{\partial X}\sin\theta + \frac{\partial W}{\partial Y}\cos\theta & \lambda_r \end{bmatrix} = \begin{bmatrix} \lambda_{cr} & -\frac{R}{2} & 0 \\ \frac{R}{2}+\epsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix}$$ i.e. $A_{11} = A_{22}$

Let $\xi = \frac{\partial W}{\partial X}\cos\theta - \frac{\partial W}{\partial Y}\sin\theta$, $\eta = \frac{\partial W}{\partial X}\sin\theta + \frac{\partial W}{\partial Y}\sin\theta$,

$cos\vartheta(\frac{\partial U}{\partial X}cos\theta - \frac{\partial U}{\partial Y}\sin\theta) - sin\theta(\frac{\partial V}{\partial X}\cos\theta - \frac{\partial V}{\partial Y}\sin\theta) = \sin\theta(\frac{\partial U}{\partial X}\sin\theta + \frac{\partial U}{\partial Y}\cos\theta) + cos\theta(\frac{\partial V}{\partial X}\sin\theta + \frac{\partial V}{\partial Y}\cos\theta)$

$(\frac{\partial U}{\partial X} - \frac{\partial V}{\partial Y})(cos\theta\ cos\theta - \sin\theta\sin\theta) - 2(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X})\sin\theta\ cos\vartheta = 0$, $\left(\frac{\partial U}{\partial X} - \frac{\partial V}{\partial Y}\right)\cos 2\theta - \left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right)\sin 2\theta = 0$ to find θ

Mathematical Foundation of Liutex

Principal Matrix and Principal Decomposition

(using a unique matrix to represent the velocity gradient tensor)

The principal tensor matrix should be

$$\nabla\vec{V} = \begin{bmatrix} \lambda_{cr} & \frac{R}{2}+\epsilon & \xi \\ -\frac{R}{2} & \lambda_{cr} & \eta \\ 0 & 0 & \lambda_r \end{bmatrix}, \quad \left(\nabla\vec{V}\right)^T = \begin{bmatrix} \lambda_{cr} & -\frac{R}{2} & 0 \\ \frac{R}{2}+\epsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix}$$

(Real Schur Decomposition equivalent to QP rotation but without coordinate rotation)

Principal decomposition

$$\nabla\vec{V} = \begin{bmatrix} \lambda_{cr} & \frac{R}{2}+\epsilon & \xi \\ -\frac{R}{2} & \lambda_{cr} & \eta \\ 0 & 0 & \lambda_r \end{bmatrix} = \begin{bmatrix} 0 & R/2 & 0 \\ -R/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_r \end{bmatrix} + \begin{bmatrix} 0 & \in & \xi \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{bmatrix} = \boldsymbol{R} + \boldsymbol{SC}+\mathbf{S}$$

$$and$$

$$\left(\nabla\vec{V}\right)^T = \begin{bmatrix} 0 & -R/2 & 0 \\ R/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_r \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \in & 0 & 0 \\ \xi & \eta & 0 \end{bmatrix} = -\boldsymbol{R} + \boldsymbol{SC}+\boldsymbol{S}^{\boldsymbol{T}}$$

Koloar 2007 and Li et al. 2014 have similar ideas to decompose the velocity gradient tensor

Kolář, V., Vortex identification: New requirements and limitations [J]. International Journal of Heat and Fluid Flow, (2007), 28(4): 638-652.

Li, Z., Zhang, X., He., F., Evaluation of vortex criteria by virtue of the quadruple decomposition of velocity gradient tensor. Acta Physics Sinica, 2014, 63(5): 054704, in Chinese

6. Divergence of a Tensor

Divergence of velocity gradient

$$\nabla \cdot \nabla\vec{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}, \ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

is a 1x3 vector and must be transposed

$$[\nabla \cdot \nabla\vec{v}]^{\boldsymbol{T}} = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{bmatrix} = \begin{bmatrix} \nabla^2 u \\ \nabla^2 v \\ \nabla^2 w \end{bmatrix} \neq \nabla \cdot \nabla\vec{v}$$

Useful for understanding fluid dynamics governing equations

We use column vector only

6. Divergence of a Tensor

Divergence of velocity gradient

$$[\nabla\cdot(\nabla\vec{v})^T]^T = \left(\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{bmatrix}\right)^{\boldsymbol{T}} = \begin{bmatrix}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x}\\ \frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}\\ \frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\\ \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\end{bmatrix}$$

$$[\nabla\cdot(\nabla\vec{v})^T]^T = \begin{bmatrix}\frac{\partial}{\partial x}tr\\ \frac{\partial}{\partial y}tr\\ \frac{\partial}{\partial z}tr\end{bmatrix} = \nabla tr = \nabla(\nabla\cdot\vec{v})$$

For incompressible flow: $[\nabla\cdot(\nabla\vec{v})^T]^T \equiv 0$

Useful for understanding fluid dynamics governing equations:

For incompressible flow:

$\nabla\cdot\nabla\vec{v}= \nabla\cdot\nabla\vec{v}+ \nabla\cdot(\nabla\vec{v})^T= \nabla\cdot[\nabla\vec{v}+(\nabla\vec{v})^T]$ - Strain (symmetric)

$\nabla\cdot\nabla\vec{v}= \nabla\cdot\nabla\vec{v} - \nabla\cdot(\nabla\vec{v})^T= \nabla\cdot[\nabla\vec{v}-(\nabla\vec{v})^T]$ - Vorticity (anti-symmetric)

Same to use strain (6 entries in N-S) or vorticity (3 entries in my new governing equations)

Conclusion

1. Vector and tensor are unique, but matrix is dependent on coordinate systems
2. Vector and tensor have dot, cross, and dyadic operations, but matrix only has addition, subtraction, multiplication, transpose, inverse.
3. Vector and tensor operators are different from matrix operators.
4. Velocity gradient formula given by Wiki and western fluid dynamics textbooks is misunderstanding and should be corrected.
5. Liutex and two other orthogonal vectors can make a rotation matrix and obtained a principal coordinate system to get the principal matrix for velocity gradient tensor which is unique
6. Cauchy-Stokes decomposition should be revisited

Reference Papers

1. Liu, C.*, Yu, Y. & Gao, Ys. Liutex based new fluid kinematics. J Hydrodyn 34, 355–371 (2022). https://doi.org/10.1007/s42241-022-0046-z
2. Liu, C.* and Liu, Z., New governing equations for fluid dynamics, AIP Advances 11, 115025 (2021); https://doi.org/10.1063/5.0074615
3. Gao, Y. and Liu, C., "Rortex and comparison with eigenvalue-based identification criteria", *Physics of Fluid* **30**, 085107 (2018).
4. Y. Wang, Y. Gao, J. Liu, C. Liu*, "Explicit formula for the Liutex vector and physical meaning of vorticity based on the Liutex-Shear decomposition", J Hydrodyn (2019). https://doi.org/10.1007/s42241-019-0032-2 with Y. Wang, Y. Gao, Y. Liu
5. Y. Gao and C. Liu*, "Letter: Rortex based velocity gradient tensor decomposition", Physics of Fluids, in processing (2019).
6. C. Liu, Y. Gao, X. Dong, J. Liu, Y. Zhang, X. Cai, L. Gui, "Third generation of vortex identification methods: Omega and Liutex/Rortex based systems", Journal of Hydrodynamics (2019), 31(2): 1-19

Thank you!